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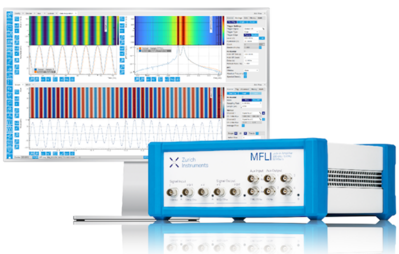

Diffusion Phenomena Associated With Surface Growth

Gabriella Bognár[1], Imre Ferenc Barna[2,a], Krisztián Hriczó[1] and László Mátyás[3]

[1]*University of Miskolc, Miskolc-Egyetemváros 3515, Hungary*
[2]*Wigner Research Centre for Physics, Konkoly–Thege Miklós út 29–33, Budapest, Hungary*
[3]*Sapientia Hungarian University of Transylvania, Department of Bioengineering, Faculty of Economics, Socio-Human Sciences and Engineering, Libertătii sq. 1, Miercurea Ciuc, Romania*

[a)]Corresponding author: barna.imre@wigner.hu

Abstract. We present a surface growth model associated with diffusion on surface. The diffusion process id described by the diffusion equation. The growth phenomena is governed by the KPZ equation with an appropriate source term.

INTRODUCTION

The diffusion is an important phenomena in nature, studied in different fields of science [1]. Mathematically it is described by the equation of diffusion. Diffusion on solid state is studied in [2] and the spreading on surfaces in [3]. New solutions, for infinite horizon, one may find in [4].

If the diffusing material spreads on a surface, a special growth phenomena may occur.

Theory and Results

At this point we present the governing equations of the process. The first equation is the diffusion equation, and the second the KPZ equation

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial t^2}, \tag{1}$$

$$\frac{\partial h(x,t)}{\partial t} = \nu\frac{\partial^2 h(x,t)}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial h(x,t)}{\partial x}\right)^2 + t^{-\frac{1}{2}} \cdot C(x,t), \tag{2}$$

where $C(x,t)$ is the distributions of the particle concentration in space and time and D is the diffusion coefficient.

The function $C(x,t)$ fulfills the necessary smoothness conditions with existing continuous first and second derivatives in respect to time and and space and from physical reasons $D > 0$. Numerous physics textbooks gives us the derivation how the fundamental (the Gaussian) solutions can be obtained e.g. [5, 6].

To derive analytic, dispersive, physically relevant solutions to (1-2) we apply two self-similar Ansätze [7] of the form:

$$C(x,t) = t^{-\alpha} g\left(\frac{x}{t^{\beta}}\right) = t^{-\alpha} g(\omega), \tag{3}$$

and

$$h(x,t) = t^{-\gamma} f\left(\frac{x}{t^{\beta}}\right) = t^{-\gamma} f(\omega). \tag{4}$$

where α, β and γ are the self-similar exponents, being real numbers describing the decay and the spreading of the solution in time and space. The shape functions $f(\omega)$ and $g(\omega)$ should have the corresponding smoothness.

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Applying Ansatz (3) and (4) to the PDE system of (1-2) the next coupled ODE system can be derived:

$$-\frac{1}{2}g - \frac{1}{2}\omega g' \quad = \quad Dg'', \tag{5}$$

$$\nu f''(\omega) + f'(\omega)\left[\frac{\omega}{2} + \frac{\lambda}{2}f'(\omega)\right] + g \quad = \quad 0. \tag{6}$$

For the self-similar exponents the following relations have to be fulfilled:

$$\alpha = \frac{1}{2}, \beta = \frac{1}{2}, \gamma = 0. \tag{7}$$

Choosing the proper initial conditions for the first diffusion equation we can get the usual Gaussian solution:

$$g = ae^{\frac{-\omega^2}{4D}}, \tag{8}$$

where a is a constant. At his point we have to emphasize that for an arbitrary real α and for $\beta = 1/2$, the solutions become much more complicated and can be expressed with the Kummer's functions. The properties of these functions one may find in NIST Handbook [8].

With this knowledge, the final form of the ODE obtained with the shape function of the KPZ equation can be given as follows:

$$\nu f''(\omega) + \frac{f'(\omega)}{2}\left[\omega + \lambda f'(\omega)\right] + ae^{\frac{-\omega^2}{4D}} = 0. \tag{9}$$

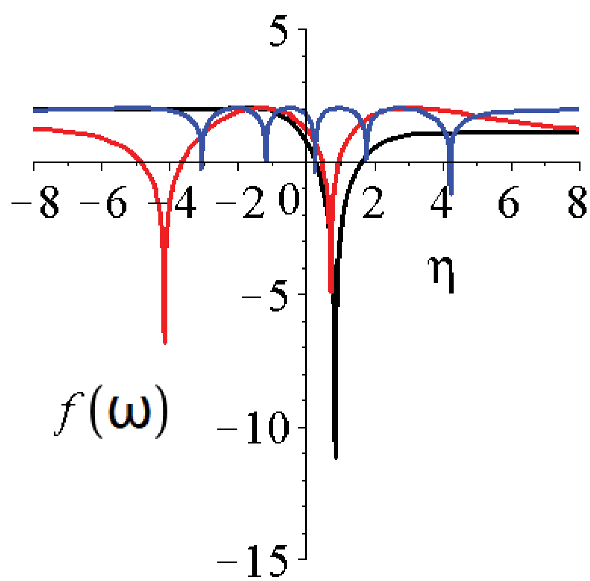

FIGURE 1. The shape functions of Eq. (10) for different values of ω for various α, λ, ν parameter sets. The black, red and blue lines belong to {1,1,1}, {4,3,5} and {8,20,4} values, respectively.

After exhaustive investigation we may state, that there is no general formula available when all the four parameters λ, η, a and D have arbitrary numerical values. Fortunately, if the constraint $4D = 2\nu$ holds the ODE has a simplified form of

$$\nu f''(\omega) + \frac{f'(\omega)}{2}\left[\omega + \lambda f'(\omega)\right] + ae^{\frac{-\omega^2}{2\nu}} = 0. \tag{10}$$

The shape functions $f(\omega)$ can be determined, from the above equation. It is interesting to see, that one may arrive to an analytic form.

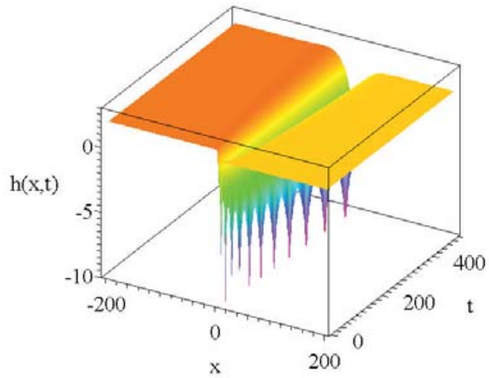

FIGURE 2. The solutions $h(x,t)$ to Eq. (2) for three specific parameter sets $\lambda = 1, \nu = 1, a = 1$. The values of integration constants are $C_1 = 1, C_2 = 2$.

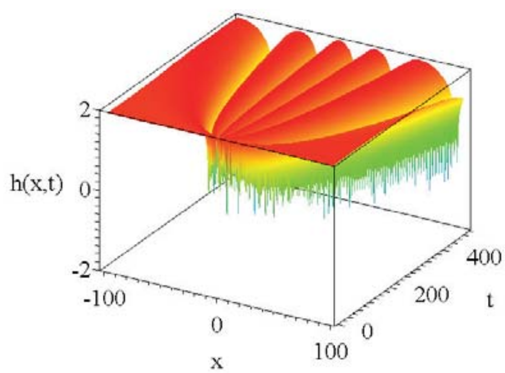

FIGURE 3. The solutions $h(x,t)$ to Eq. (2) for the parameters $\lambda = 20, \nu = 4, a = 8$ and the integration constants $C_1 = 1, C_2 = 2$.

Using the Maple 12 mathematical software package the next closed expression for the solution can be derived

$$f(\omega) = -\frac{\nu}{\lambda} \ln \left(1 + \left[\tan \left\{ \frac{\sqrt{2\lambda a \pi} \cdot erf(\frac{\omega}{2\sqrt{\nu}}) + 2C_1\sqrt{\nu}}{2\sqrt{\nu}} \right\} \right]^2 \right) + C_2, \tag{11}$$

where erf denotes the error function. To learn more mathematical properties consult the handbook of [8]. Note, that the expression depends on all three strength parameters λ, ν and a, where C_1, C_2 are the free integration constants. Figure 1 presents various solutions for different values of $\{a, \lambda, \nu\}$. The properties of these curves are far from being trivial. It is important to emphasize that due to the properties of the tan function the solutions may go to infinity with infinite first derivatives at given points. The presented plots have however finite singularities, this is due to the Maple 12 software.

Regarding the shape of the surface, it can be evaluated by inserting the source term in the KPZ equation. This yields an interesting shape function $h(x,t)$, which depends on the parameters presented above.

The results show, that depending on the argument of logarithm, one may find different shapes of the solution. This means that from few to more local maxima can be observed depending on the corresponding argument.

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