

Diffusion Cascades and Mutually Coupled Diffusion Processes

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Abstract: In this paper, we define and investigate a system of coupled regular diffusion equations in which each concentration acts as a driving term in the next diffusion equation. Such systems can be understood as a kind of cascade process which appear in different fields of physics like diffusion and reaction processes or turbulence. As a solution, we apply the time-dependent self-similar Ansatz method, the obtained solutions can be expressed as the product of a Gaussian and a Kummer's function. This model physically means that the first diffusion works as a catalyst in the second diffusion system. The coupling of these diffusion systems is only one way. In the second part of the study we investigate mutually coupled diffusion equations which also have the self-similar trial function. The derived solutions show some similarities to the former one. To make our investigation more complete, different kinds of couplings were examined like the linear, the power-law, and the Lorentzian. Finally, a special coupling was investigated which is capable of describing isomerization with temporal decay.

Keywords: diffusion; self-similar solution

MSC: 35K57; 35Cxx; 60J60



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1. Introduction

Transport processes have gained significant interest both for science and in engineering applications. The simplest one being heat conduction in solids or the regular diffusion of particles. Without additional couplings to other processes, these two related phenomena alone are well-understood and do not attract much interest. (The literature on diffusion (or heat conduction) is immense; therefore, we only mention some recent monographs and studies [1–10]).

So, more elaborate and compound processes have to be coupled together and must be investigated simultaneously to match the logic of today's science. In other words the objects examined within science are becoming more and more complex. One strategy is to couple as many elementary processes together as possible up to the reasonable limit. A good example of this method is the Fermi–Pasta–Ulam–Tsingou model, where a large number of weakly linear oscillators were coupled, examined and finally highly non-linear effects like deterministic chaos or solitons were found as relevant new types of physics [11,12]. We are now following this path also. In the following paper, we define and analyze models which are systems of coupled regular diffusion equations. Firstly, we define a model where the first diffusion propels the second one and the second serves as source or driving term in the third one and so on. We call this model the diffusion cascade. (This phrase was already used in the scientific literature which we mention in detail in the next section). In our former studies, we analyzed the regular diffusion equation with the self-similar

Ansatz in depth and found new analytic solutions which are available on the whole regular axis instead of the usual Fourier ones which are defined on a finite interval only [13,14]. Recently, we also published analytic solutions to the non-linear diffusion equation [15]. As will be seen, the derived solutions become more and more compound, containing more and more terms, and showing a larger and larger number of oscillations and internal structure. Numerical methods for solving diffusion equations have also improved a lot in recent years [16–18].

There is a high degree of generalization possible, but we only mention one model. Cascade processes are well-known in many fields of science like high-energy particle collisions with each other or with ordinary matter [19,20], in turbulence [21], in biochemistry [22], and in ecology [23]. Cascade processes have strong connection to stochastic processes [24] as well. The phrase diffusion cascade was defined and used to describe Spreading Phenomena in Blog Network Communities [25] in 2012 and later by [26].

In the second part of the study we present a type of mutually coupled diffusion, or rather reaction–diffusion equations [27], which also have closed-form solutions derived with the self-similar Ansatz. Such models exist in different fields of science like mathematical biology [28], pattern formation [29], or even in plasma physics [30,31]. The possibility of describing isomerization is important for oil refining products— in which normal paraffins may be converted to isoparaffins [32]— or in the design of less expensive and flexible electronic devices [33].

The paper presents certain novel functions as solutions for the cascading diffusion system, the two-component time decaying isomerization, and the two-component diffusion system with cross diffusion, where the system is not necessarily constrained by a finite boundary.

General Aspects of Diffusion for Infinite Horizon

Before we present our results, we briefly sketch our solution method. We start with the usual diffusion (or heat conduction equation) for one Cartesian coordinate:

$$\frac{\partial C(x, t)}{\partial t} = D \frac{\partial^2 C(x, t)}{\partial x^2}, \tag{1}$$

with the self-similar Ansatz [34–36] in the form of $C_1(x, t) = t^{-\alpha} \cdot f\left(\frac{x}{t^\beta}\right)$, where $\eta = \frac{x}{t^\beta}$ is the reduced variable. (This Ansatz can be generalized even for two Cartesian variables [37]). After the usual algebraic steps we arrive to the ODE of

$$-\alpha f - \frac{\eta}{2} f' = D f'', \tag{2}$$

with the constraint of $\beta = 1/2$ and α which can be arbitrary or real. We can derive the most general solution in the form of

$$f(\eta) = e^{-\frac{\eta^2}{4D}} \cdot \eta \cdot \left(c_1 M\left[\alpha - 1, \frac{3}{2}, \frac{\eta^2}{2D}\right] + c_2 U\left[\alpha - 1, \frac{3}{2}, \frac{\eta^2}{2D}\right] \right), \tag{3}$$

using the original variables, and the final concentration reads as follows:

$$C_\alpha(x, t) = e^{-\frac{x^2}{4Dt}} \cdot \left(\frac{x}{t^\beta}\right) \cdot \left(c_1 M\left[\alpha - 1, \frac{3}{2}, \frac{x^2}{2Dt}\right] + c_2 U\left[\alpha - 1, \frac{3}{2}, \frac{x^2}{2Dt}\right] \right). \tag{4}$$

where M and U are the Kummer’s functions [38], and c_1 and c_2 are the real integration constants. The properties of Equation (3) were exhaustively studied in our former studies. Depending on the numerical values of the first parameter, the Kummer’s functions can have four different kinds of properties, which were also detailed in our earlier publications. We showed that, with a positive integer αs , the $\eta \cdot M(, , \eta^2)$ function is reduced to finite odd order polynomials, and for a positive half-integer, αs the $\eta \cdot U(, , \eta^2)$ function is reduced

to even-order polynomials [39]. Such solutions have power-law temporal decay, which is physically required. As an example, we define the first three odd and even polynomials:

$$\eta \cdot M\left(1 - \alpha, \frac{3}{2}, \frac{\eta^2}{2D}\right), \quad \alpha = 1, \quad \eta, \tag{5}$$

$$\eta \cdot M\left(1 - \alpha, \frac{3}{2}, \frac{\eta^2}{2D}\right), \quad \alpha = 2, \quad \eta - \frac{\eta^3}{6D}, \tag{6}$$

$$\eta \cdot M\left(1 - \alpha, \frac{3}{2}, \frac{\eta^2}{2D}\right), \quad \alpha = 3, \quad \eta - \frac{\eta^3}{3D} + \frac{\eta^5}{60D}, \tag{7}$$

and

$$\eta \cdot U\left(1 - \alpha, \frac{3}{2}, \frac{\eta^2}{2D}\right), \quad \alpha = 1/2, \quad 2\sqrt{D}, \tag{8}$$

$$\eta \cdot U\left(1 - \alpha, \frac{3}{2}, \frac{\eta^2}{2D}\right), \quad \alpha = 3/2, \quad -\sqrt{D} + \frac{1}{2\sqrt{D}}\eta^2, \tag{9}$$

$$\eta \cdot U\left(1 - \alpha, \frac{3}{2}, \frac{\eta^2}{2D}\right), \quad \alpha = 5/2, \quad \frac{3\sqrt{D}}{2} - \frac{3\eta^2}{2\sqrt{D}} + \frac{\eta^4}{8D^{3/2}}. \tag{10}$$

These will be important in the following analysis. Unfortunately, Kummer’s function with the quadratic argument cannot form a complete set of orthogonal functions, and therefore cannot be used for a proper series expansion.

2. Theory and Results

In the main part of our study we define two types of coupled diffusion equations, solve them, and analyze the obtained results.

2.1. Diffusion Cascades

We introduce our iterative PDE system of

$$\frac{\partial C_1(x, t)}{\partial t} = D_1 \frac{\partial^2 C_1(x, t)}{\partial x^2}, \tag{11}$$

$$\frac{\partial C_2(x, t)}{\partial t} = D_2 \frac{\partial^2 C_2(x, t)}{\partial x^2} + a_{12} \frac{C_1(x, t)}{t}, \tag{12}$$

$$\frac{\partial C_3(x, t)}{\partial t} = D_3 \frac{\partial^2 C_3(x, t)}{\partial x^2} + a_{13} \frac{C_1(x, t)}{t} + a_{23} \frac{C_2(x, t)}{t}, \tag{13}$$

$$\dots = \dots \tag{14}$$

$$\dots = \dots \tag{15}$$

where the D_i s are the diffusion coefficients which are all positive real numbers, the a_{ij} s are the coupling constants, and the C_i s are the investigated concentration functions with the satisfactory smoothness. To obtain analytic solutions, as a first step, we truncate the result after the the second equation. This means that we consider only Equations (11) and (12). Such a situation describes the diffusion of two components, where the first component acts as a catalyst in the system, facilitating the production of the second component. Recent results from catalytic diffusion have been published in [40].

We apply the self-similar Ansatz

$$C_1(x, t) = t^{-\alpha} \cdot f\left(\frac{x}{t^\beta}\right), \quad C_2(x, t) = t^{-\gamma} \cdot g\left(\frac{x}{t^\beta}\right), \tag{16}$$

where $\eta = \frac{x}{t^\beta}$ is the reduced variable. All three concentrations have the form of Equation (4).

Consider Equation (12) first with a finite odd $\alpha = 2$ source term and derive the corresponding ODE

$$-\gamma g - \frac{\eta}{2} g' = D g'' + a_{12} e^{\frac{\eta^2}{4D}} \left(\eta - \frac{\eta^3}{6D} \right). \tag{17}$$

The solution is

$$g(\eta) = e^{-\frac{\eta^2}{4D}} \cdot \eta \cdot \left(c_1 M \left[1 - \gamma, \frac{3}{2}, \frac{\eta^2}{4D} \right] + c_2 U \left[1 - \gamma, \frac{3}{2}, \frac{\eta^2}{4D} \right] \right) - \frac{a_{12} \eta e^{-\frac{\eta^2}{4D}} (D - 1/6\eta^2)}{D(\gamma - 2)}, \tag{18}$$

so we essentially return to the usual Gaussian times Kummer’s function minus the original source or driving term. This is easy to see on this ODE, but far from being simple in the PDE system level in Equations (11)–(13). Note, that $\gamma \neq 2$ which means that only a different kind of solution can form a new linear combination. Figure 1a shows the $g(\eta)$ shape functions for different parameter sets. Figure 1b, however, presents a final possible $C_2(x, t)$

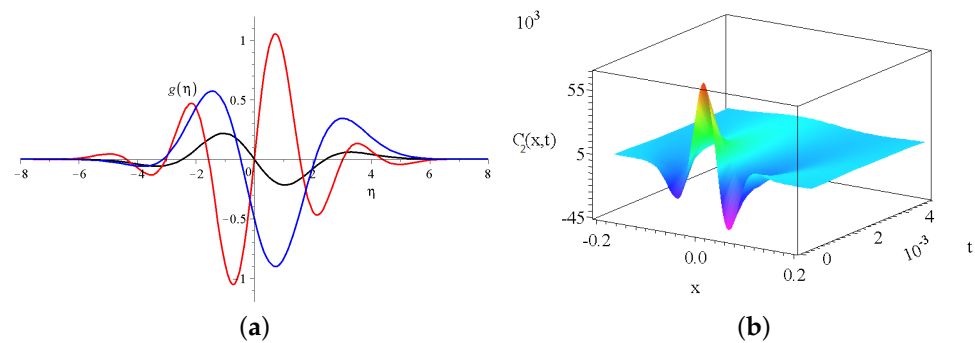


Figure 1. (a) The $g(\eta)$ shape functions of Equation (18) for three different parameter sets $\{D, c_1, c_2, a_{12}, \gamma\}$. The black (just the source term), red (with Kummer’s M), and blue (with Kummer’s U) curves are for $\{1, 0, 0, 1, 2\}$, $\{1, 2, 0, 1, 5\}$, and $\{1, 0, 1, 0.5, 1.5\}$, respectively. (b) The $C_2(x, t)$ is for the parameter set of $\{1, 0, 1, 0.5, 1.5\}$.

Our experience shows that, for a closed solution, both diffusion coefficients should be equal to each other. So different diffusion processes with different coefficients cannot be coupled together with the self-similar Ansatz.

Fixing the numerical value of γ and c_1, c_2 , even and odd solutions can be defined. Due to the linearity of Equation (17), it is easy to see that any kind of Gaussian-times-finite polynomial source term can be considered and an analytic solution can be derived. The opposite argument is also true, namely that a complicated solution with numerous terms one after another in a diffusion equation can be considered as a finite series of coupled diffusion processes in which everything in the previous solution is the source of the next.

2.2. Coupled Diffusion Equations

We defined a cascades as a one-way coupling of equations and the corresponding processes together. The number of equations (considering different modes) can be increased as desired. Here, we consider mutually coupled diffusion equations, with specific coupling terms, which can be called a reaction–diffusion equation system (just to define the used phrases correctly, diffusion equations have no extra term over the second-order spatial derivative, drift–diffusion equations have first-order spatial derivative terms, and reaction–diffusion equations have extra source terms containing no derivation. Using this logic, drift–reaction–diffusion equations can also be considered as the most general case). We start with the simplest case, where only two equations are coupled:

$$\begin{aligned} \frac{\partial C_1(x, t)}{\partial t} &= D_1 \frac{\partial^2 C_1(x, t)}{\partial x^2} + F(C_1(x, t), C_2(x, t)), \\ \frac{\partial C_2(x, t)}{\partial t} &= D_2 \frac{\partial^2 C_2(x, t)}{\partial x^2} + G(C_1(x, t), C_2(x, t)), \end{aligned} \tag{19}$$

where $F()$ and $G()$ are, in principle, any kind of continuous function. There are numerous physically relevant but complicated couplings available between two diffusion equations like the complex Ginsburg–Landau [41,42] or the Gross–Pitaevskii [43,44] equations, after separating the real and the complex parts of the solution. We will see in the next section that the self-similar Ansatz of Equation (16) radically reduces the possible couplings between the two processes. The simplest case is when

$$F(C_1, C_2) = \frac{a \cdot C_2}{t}, \quad G(C_1, C_2) = \frac{b \cdot C_1}{t}, \tag{20}$$

In the reduced ODE system we consider that $D_1 = D_2$, however, the two external self-similar exponents can still be different $\alpha \neq \gamma$ and $\beta = 1/2$. The solutions are quite cumbersome but the next two abbreviations, $p = 1 - \frac{\gamma}{2} - \frac{\alpha}{2}$ and $q = (\sqrt{4ab - 2\alpha\gamma + \gamma^2 + \alpha^2})/2$, can be given in the closed form in a compact way:

$$f(\eta) = \eta \cdot e^{-\frac{\eta^2}{4D}} \cdot \left(c_1 M \left[p - q, \frac{3}{2}, \frac{\eta^2}{4D} \right] + c_2 M \left[p + q, \frac{3}{2}, \frac{\eta^2}{4D} \right] + c_3 U \left[p - q, \frac{3}{2}, \frac{\eta^2}{4D} \right] + c_4 U \left[p + q, \frac{3}{2}, \frac{\eta^2}{4D} \right] \right), \tag{21}$$

$$g(\eta) = \frac{\eta}{2a} \cdot e^{-\frac{\eta^2}{4D}} \cdot \left(c_1 M \left[p - q, \frac{3}{2}, \frac{\eta^2}{4D} \right] [\gamma - \alpha + 2q] + c_2 M \left[p + q, \frac{3}{2}, \frac{\eta^2}{4D} \right] [\gamma - \alpha - 2q] + c_3 U \left[p - q, \frac{3}{2}, \frac{\eta^2}{4D} \right] [\gamma - \alpha + 2q] + c_4 U \left[p + q, \frac{3}{2}, \frac{\eta^2}{4D} \right] [\gamma - \alpha - 2q] \right), \tag{22}$$

where c_1, c_2, c_3 , and c_4 are the normal real integration constants. At first glance, Equations (21) and (22) look difficult to analyze and understand. The key parameter which essentially defines the global properties of the solutions is the first parameter of both Kummer’s functions which we notate now as $p \pm q$. Let us analyze this parameter; for us, the finite polynomial solutions of both Kummer’s function are relevant to Equations (5)–(10).

The formula is symmetric in the interchange of the parameters a, b and in α, γ . It is clear that if $a \cdot b > 0$ then the expression under the square root is always real for any α or γ because $(\alpha - \gamma)^2 > 0$. For $\alpha = \gamma = 1$, we obtain the simplest form of the solution

$$f() \propto c_1 M(-\sqrt{ab}, ,) + c_2 M(+\sqrt{ab}, ,) + c_3 U(-\sqrt{ab}, ,) + c_4 U(+\sqrt{ab}, ,). \tag{23}$$

We know from previous work that if first parameter of the Kummer’s M function is a negative integer, we obtain a finite polynomial resulting in a solution that oscillates and decays in time and space. This can be fulfilled with (1) if $a = b \in \mathbb{N}$ or (2) if a and b are square numbers.

Additional constraints which should be fulfilled are $c_2 = c_3 = c_4 = 0$. Other possible physically relevant solutions are when the first parameter in the Kummer’s U functions is a negative half-integer, so the square of $a \cdot b$ should be a half integer which can be fulfilled for infinite fixed a and b pairs. In general we can say that, for any kind of real α and β , we can find a and b pairs that have a negative integer as the first parameter for the Kummer’s M or a negative half integer for the Kummer’s U function.

Figure 2 presents a shape function $f(\eta)$ and the corresponding concentration $C_1(x, t)$ distribution for Kummer’s M functions for a physically relevant temporal and spatial decay-property.

In the opposite case ($a \cdot b < 0$), two possibilities can happen:

- The first parameter of the Kummer’s functions $p \pm q$ is purely imaginary $\in i\mathbb{R}$ when $1 - \frac{\alpha}{2} - \frac{\gamma}{2} = 0$. In the simplest and symmetrical case when $\alpha = \gamma = 1/2$ due to the Gaussian prefactor, both the real and the imaginary parts of the solutions containing either $M(,,)$ or $U(,,)$ functions have decaying properties for asymptotic temporal and spatial variables independently from the sign of the parameter. Figure 3 presents

the imaginary part of such a shape function and the imaginary part of the final concentration for a given set of parameters. For an asymmetric general case, when α or γ becomes negative then the corresponding $C_{1,2}(x, t)$ solution becomes divergent at infinite times due to the temporal prefactors of t^α or t^δ .

The other diffusion functions, however, have the proper decaying property.

- $p \pm q \in \mathbb{Z}$ is a general complex number (and not purely imaginary) that occurs when $1 - \frac{\gamma}{2} - \frac{\alpha}{2} \neq 0$. To obtain solutions with the proper asymptotic, the real part of the parameter should be negative and the corresponding self-similar exponents which are responsible for the decay should be positive (α, γ). Unfortunately, for a quite general set of parameters, all the terms of Equations (21) and (22) must be studied with great care.

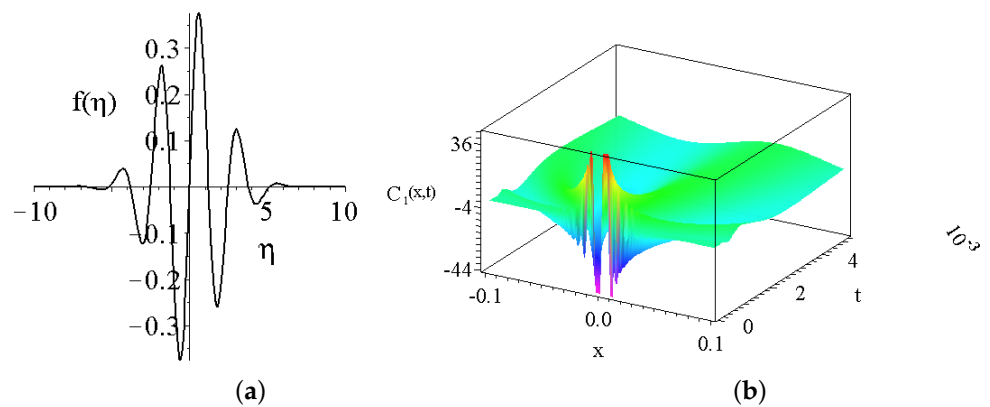


Figure 2. The shape function $f(\eta)$ of Equation (22) and the final concentration $C_1(x, t)$ for the set of parameters $D = 1, c_1 = 1, c_2 = c_3 = c_4 = 0, a = 4, b = 9, \alpha = 1/2, \gamma = 1$. (a) represents $f(\eta)$ and (b) shows $C_1(x, t)$, respectively.

In both complex cases, the Kummer’s functions become an infinite complex polynomial in η .

Except for the forced harmonic oscillator problem (due to the Euler formula), the complex solutions to real ODEs are hard to understand and explain. Generally, the following statement is true: separately, both the real and the imaginary parts of the resulting functions are solutions for the corresponding ODE.

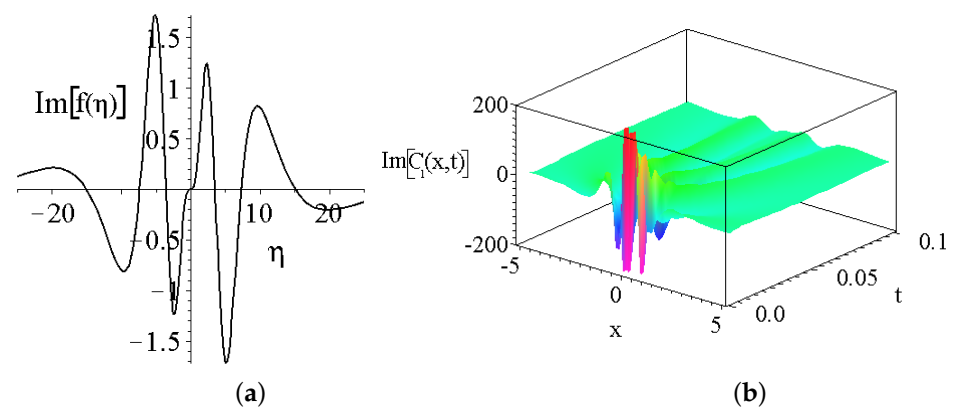


Figure 3. The imaginary part of the shape function $f(\eta)$ in Equation (22) and the imaginary part of the final concentration $C_1(x, t)$ for the set of parameters $D = 1, c_2 = 1, c_1 = c_3 = c_4 = 0, a = -4, b = 1, \alpha = 1, \gamma = 1$. (a) represents $Im[f(\eta)]$ and (b) shows $Im[C_1(x, t)]$, respectively.

For completeness, we have to mention that closed-form solutions exist for some special forms of coupling terms like

$$F(C_1, C_2) = \frac{a \cdot C_2 \cdot \eta^n}{t}, \quad G(C_1, C_2) = \frac{b \cdot C_1 \cdot \eta^n}{t}, \quad (24)$$

if and only if $n = 1, 2, -1, -2, \alpha = \gamma$ and $D_1 = D_2$, the solutions look similar and can be expressed with Kummer’s, Whittaker’s [38], or the Heun [38,45] functions.

If $\alpha = \gamma$ and $D_1 = D_2$, even for the Lorentzian function

$$F(C_1, C_2) = \frac{a \cdot C_2}{t(1 + \eta^2)}, \quad G(C_1, C_2) = \frac{b \cdot C_1}{t(1 + \eta^2)}, \tag{25}$$

Maple 12 can evaluate analytical solutions containing the Heun Confluent functions. We think that these kind of coupling source terms have little practical interest hence we skip their detailed analysis here. We may go further and try to couple three or even more diffusion equations together with different couplings terms. Our experience shows that, in general, even for three-part coupled diffusion equations the most general self-similar solutions and formulas become tediously long and opaque. In some special cases (when all remaining free self-similar exponents are set to 1/2) and the first two diffusion equations are coupled and the sum of the two diffusion equations is considered as a source term in the third one, then the solution remains transparent. A detailed analysis of this could be the topic of a subsequent study.

2.3. The Time-Decaying Isomerization

In this Section, we study the following coupled equations which describe the isomerization process [46,47]. In chemistry, isomerization is the process in which a polyatomic ion, molecule, or molecular fragment is transformed into an isomer with a different chemical structure. The general relevant equations are presented in [48]. Relative to that system, in this subsection, we neglect the cross diffusion and we have made a slight modification regarding the reaction rate. At this point we consider a reaction where the reaction rate changes in time. Here, we follow a decaying reaction rate k/t . With spatial extension, the reaction–diffusion equations read

$$\begin{aligned} \frac{\partial C_1(x, t)}{\partial t} &= D_1 \frac{\partial^2 C_1(x, t)}{\partial x^2} - \frac{k}{t}(C_1(x, t) - C_2(x, t)), \\ \frac{\partial C_2(x, t)}{\partial t} &= D_2 \frac{\partial^2 C_2(x, t)}{\partial x^2} + \frac{k}{t}(C_1(x, t) - C_2(x, t)), \end{aligned} \tag{26}$$

where k/t is a decaying reaction rate. Applying the transformation (16) and as a first step taking $D_1 = D_2 = D$, one arrives at the following coupled ODE system:

$$\begin{aligned} -\alpha f(\eta) - \frac{1}{2}\eta \frac{df}{d\eta} &= D \frac{\partial^2 f}{\partial \eta^2} - k(f(\eta) + g(\eta)), \\ -\gamma g(\eta) - \frac{1}{2}\eta \frac{dg}{d\eta} &= D \frac{\partial^2 g}{\partial \eta^2} + k(f(\eta) - g(\eta)). \end{aligned} \tag{27}$$

The solutions to these equations—with the following two notations $p = 1 - \frac{\gamma}{2} - \frac{\alpha}{2}$ and $q' = (\sqrt{4k^2 - 2\alpha\gamma + \gamma^2 + \alpha^2})/2$ —are

$$f(\eta) = \eta e^{-\frac{\eta^2}{4D}} \left(c_1 M \left[p - q' + k, \frac{3}{2}, \frac{\eta^2}{4D} \right] + c_2 M \left[p + q' + k, \frac{3}{2}, \frac{\eta^2}{4D} \right] + c_3 U \left[p - q' + k, \frac{3}{2}, \frac{\eta^2}{4D} \right] + c_4 U \left[p + q' + k, \frac{3}{2}, \frac{\eta^2}{4D} \right] \right), \tag{28}$$

$$g(\eta) = \frac{\eta}{2k} \cdot e^{-\frac{\eta^2}{4D}} \cdot (c_1 M \left[p - q' + k, \frac{3}{2}, \frac{\eta^2}{4D} \right] [\gamma - \alpha + 2q'] + c_2 M \left[p + q' + k, \frac{3}{2}, \frac{\eta^2}{4D} \right] [\gamma - \alpha - 2q'] + c_3 U \left[p - q' + k, \frac{3}{2}, \frac{\eta^2}{4D} \right] [\gamma - \alpha + 2q'] + c_4 U \left[p + q' + k, \frac{3}{2}, \frac{\eta^2}{4D} \right] [\gamma - \alpha - 2q']). \tag{29}$$

The parameter dependence is very similar to the former model, with some differences. Here, we study certain specific cases. If $\alpha = \gamma = 1, p = 0$ and $q = k$. In this case, the Kummer functions have the following behavior

$$M \left[p - q' + k, \frac{3}{2}, \frac{\eta^2}{4D} \right] = M \left[0, \frac{3}{2}, \frac{\eta^2}{4D} \right] = 1 \tag{30}$$

$$U \left[p - q' + k, \frac{3}{2}, \frac{\eta^2}{4D} \right] = U \left[0, \frac{3}{2}, \frac{\eta^2}{4D} \right] = 1, \tag{31}$$

in which $M \left[2k, \frac{3}{2}, \frac{\eta^2}{4D} \right]$ is divergent and $U \left[2k, \frac{3}{2}, \frac{\eta^2}{4D} \right]$ has a less usual behavior for a large η . This means that we can choose the constants c_1 and c_3 as the initial and boundary conditions required, and c_2 and c_4 should be set to zero to avoid divergences.

The next situation is when α and γ is larger than one. A specific case is $\alpha = \gamma = 2$. Note that these self-similar parameters cause a singularity in the origin ($t = x = 0$). The shape function f and the related solution $C(x, t)$ for this specific case can be seen in Figure 4.

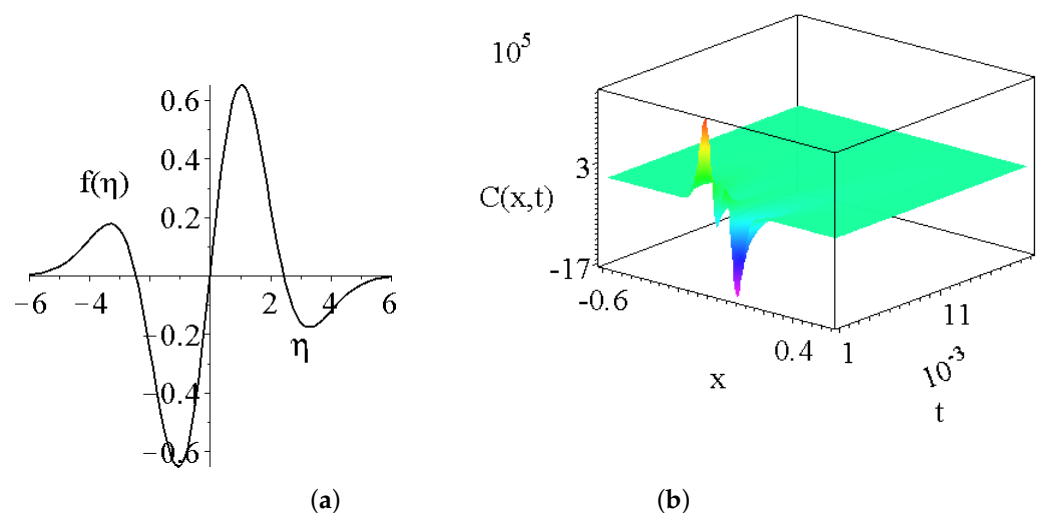


Figure 4. The shape function $f(\eta)$ (a) and $C(x, t)$ (b) of Equation (28) for the set of parameters $D = 1, c_1 = 1, c_2 = c_3 = c_4 = 0, k = 1, \alpha = 2, \gamma = 2$, respectively.

For a larger $\alpha = \gamma$, the first parameters in both Kummer’s functions may be negative which gives further relevant oscillatory and decaying solutions like Equations (5)–(10).

2.4. Coupled Diffusion Equations Obtained via the Flick’s Law

In the case of binary diffusion, in certain cases, the cross diffusion term may be important. In this situation, the diffusion of one of the components may involve the transport of the other one too [49]. This situation is characterized by nonzero cross diffusion coefficients. Correspondingly, we investigate the diffusion systems in which the coupling comes from the Fick’s law, like

$$\begin{aligned} q_1 &= D_{11}\nabla C_1 + D_{12}\nabla C_2, \\ q_2 &= D_{21}\nabla C_1 + D_{22}\nabla C_2. \end{aligned} \tag{32}$$

At this point, we should mention, that the stability condition for this system of coupled equations is $D_{11} \cdot D_{22} - D_{12} \cdot D_{21} > 0$ [49]. We restrict our study only for cases when this stability condition holds.

With the usual substitution into the corresponding continuity equations we arrive to the coupled diffusion equations of:

$$\begin{aligned} \frac{\partial C_1(x, t)}{\partial t} &= D_{11} \frac{\partial^2 C_1(x, t)}{\partial x^2} + D_{12} \frac{\partial^2 C_2(x, t)}{\partial x^2}, \\ \frac{\partial C_2(x, t)}{\partial t} &= D_{21} \frac{\partial^2 C_1(x, t)}{\partial x^2} + D_{22} \frac{\partial^2 C_2(x, t)}{\partial x^2}, \end{aligned} \tag{33}$$

using our standard Ansatz of $C_1 = t^{-\alpha} f(\eta)$ and $C_2 = t^{-\delta} g(\eta)$, the next coupled ODE system can be easily derived:

$$\begin{aligned} -\alpha f - \frac{\eta}{2} f' &= D_{11} f'' + D_{12} g'', \\ -\delta g - \frac{\eta}{2} g' &= D_{21} f'' + D_{22} g'', \end{aligned} \tag{34}$$

with the constraint of $\alpha = \delta \in \mathbb{R}$. We found that carrying out an exhaustive analysis with Maple 12 was tedious and only provided closed analytic solutions for the special sets parameters of like $\{D_{11} = D_{22}, D_{12} = D_{21}\}$ where $\alpha = 0, \pm 1/2, \pm 1$. (For some other α s, e.g., one-third or one-quarter, formulas exist but contain additional formal integration operations). The most important but tedious general formula for $\alpha = 1/2$ can be found in Appendix A of the present study. Here, we just give the result for the set of parameters $\alpha = 1/2, D_{11} = 3, D_{12} = 1/3$,

$$\begin{aligned} f(\eta) = g(\eta) &= c_1 e^{-\frac{3\eta^2}{32}} + c_2 e^{-\frac{3\eta^2}{40}} - 5c_3 \operatorname{erf}\left(\frac{I\sqrt{6}\eta}{8}\right) \sqrt{6} e^{-\frac{3\eta^2}{32}} + \\ &2c_3 \operatorname{erf}\left(\frac{I\sqrt{30}\eta}{20}\right) \sqrt{30} e^{-\frac{3\eta^2}{40}} - 2c_4 \operatorname{erf}\left(\frac{I\sqrt{6}\eta}{8}\right) \sqrt{6} e^{-\frac{3\eta^2}{32}} + c_4 \operatorname{erf}\left(\frac{I\sqrt{30}\eta}{20}\right) \sqrt{30} e^{-\frac{3\eta^2}{40}}, \end{aligned} \tag{35}$$

where erf() means the error function [38]. Figure 5 shows the imaginary part of the shape function of Equation (35) and the final $C_1(x, t)$ as well. Here, we find the usual quick temporal and spatial dependence.

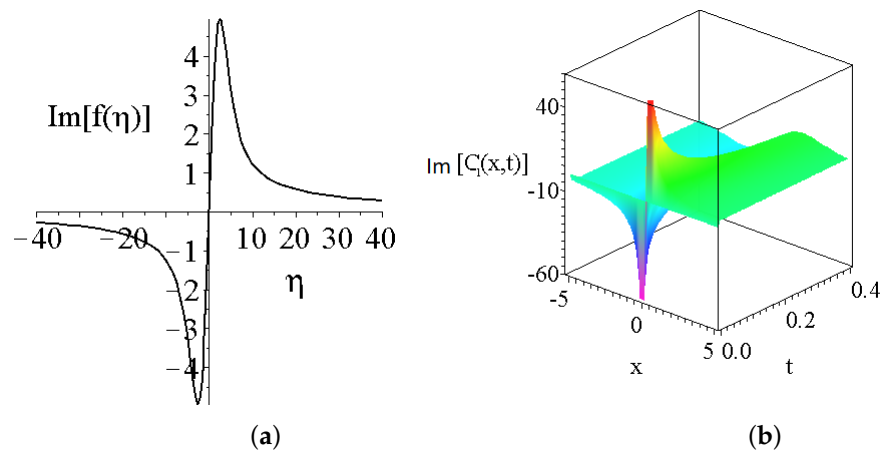


Figure 5. (a) The imaginary part of the Equation (35) for of $\alpha = 1/2, D_{11} = 3, D_{12} = 1/3$, and for $c_1 = c_2 = c_3 = 0, c_4 = 1$ (b) the $\Im[C_1(x, t)]$ for the same set of parameters, respectively.

3. Summary and Outlook

We investigated different kinds of coupled diffusion equations with the self-similar Ansatz. As an initial system, we defined one-way coupling between numerous regular diffusion equations, where the later equation has the former one as a source. We found that the Gaussian function times a Kummer’s function with an arbitrary order can be a source term with numerous quickly decaying oscillations. This property can understood from the linearity of the diffusion equation.

Secondly, we investigated two mutually coupled diffusion equations and found analytic solutions for some different kind of couplings. The linear, the power-law, and the Lorentzian couplings were examined. The linear coupling gave as the usual Gaussian times the Kummer’s function solution. However, the four independent parameters expand our understanding of the solutions. Real and complex solution can appear. It is worth mentioning that some sets of parameters give us physically relevant finite solutions with proper asymptotic decays at infinite times and as a final third model we analyzed a time-decaying isomerization process where the derived results are similar to the coupled diffusion case. The parameter dependence were exhaustively analyzed in all investigated models.

In principle we could define a further model where the diffusion coefficient is temperature dependent.

In the future we plan to investigate drift–diffusion equations for semiconductors, which couple the concentrations of electrons and holes [50].

A second possible form of investigation is to consider the traveling wave Ansatz $C(x, t) = f(x \pm ct)$ and create some kind of sinusoidal coupling between the diffusion fields. The solutions will be proportional to the Mathieu functions [38,51].

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Appendix A

For a better transparency we introduce the following four abbreviations:

$$D_p = \sqrt{-\frac{1}{D_1 + D_2}}, \quad D_m = \sqrt{-\frac{1}{D_1 - D_2}}, \quad d_p = -\frac{1}{4(D_1 + D_2)}, \quad d_m = -\frac{1}{4(D_1 - D_2)}, \quad (A1)$$

when $\alpha = \frac{1}{2}$ we get:

$$f(\eta) = g(\eta) = \frac{1}{2D_p D_m (D_1^2 - D_2^2)} \left(2c_1 D_2^2 D_m D_p e^{d_m \eta^2} - 2c_1 D_1^2 D_m D_p e^{d_m \eta^2} - \right. \\ \left. - 2c_2 D_2^2 D_p D_m e^{d_p \eta^2} + 2c_2 D_1^2 D_p D_m e^{d_p \eta^2} + \right. \\ \left. c_3 \sqrt{\pi} e^{d_m \eta} \operatorname{erf} \left[\frac{D_m \eta}{2} \right] D_p [D_1 + D_2] + c_3 \sqrt{\pi} e^{d_p \eta} \operatorname{erf} \left[\frac{D_p \eta}{2} \right] D_m [D_1 - D_2] + \right. \\ \left. + 2c_4 \sqrt{\pi} e^{d_m \eta} \operatorname{erf} \left[\frac{D_m \eta}{2} \right] D_2 D_p [D_1^2 - D_2^2] + 2c_4 \sqrt{\pi} e^{d_p \eta} \operatorname{erf} \left[\frac{D_p \eta}{2} \right] D_2 D_m [D_1^2 - D_2^2] \right). \quad (A2)$$

References

- Ghez, R. *Diffusion Phenomena*; Dover Publication Inc.: Mineola, NY, USA, 2001.
- Vogel, G. *Adventure Diffusion*; Springer: Berlin/Heidelberg, Germany, 2019.
- Bird, R.B. *Transport Phenomena*; John Wiley & Sons, Inc.: Hoboken, NJ, USA, 2002.
- Lienemann, J.; Yousefi, A.J.K. Dimension Reduction of Large-Scale Systems. *Lect. Notes Comput. Sci. Eng.* **2005**, *45*, 327.
- Thambynayagam, R. *The Diffusion Handbook: Applied Solutions for Engineers*; McGraw-Hill: New York, NY, USA, 2011.
- Benett, T. *Transport by Advection and Diffusion: Momentum, Heat and Mass Transfer*; John Wiley & Sons: Hoboken, NJ, USA, 2013.
- Latif, M. *Heat Convection*; Springer: Berlin/Heidelberg, Germany, 2009.
- Benjan, A. *Convection and Heat Transfer*; Wiley: Hoboken, NJ, USA, 2004.
- Ashan, A. *Convection and Conduction Heat Transfer*; Intec: London, UK, 2011.
- Mátyás, L.; Tél, T.; Vollmer, J. Multibaker map for shear flow and viscous heating. *Phys. Rev. E* **2001**, *64*, 056106. [[CrossRef](#)] [[PubMed](#)]
- Fermi, E.; Pasta, P.; Ulam, S.; Tsingou, M. *Studies of Nonlinear Problems*; I Los Alamos Report LA-1940 (1955), Published Later in Collected Papers of Enrico Fermi; Segré, E., Ed.; University of Chicago Press: Chicago, IL, USA, 1965. [[CrossRef](#)]
- Bambusi, D.; Ponso, A. On metastability in FPU. *Comm. Math. Phys.* **2006**, *264*, 539. [[CrossRef](#)]
- Mátyás, L.; Barna, I.F. General Self-Similar Solutions of Diffusion Equation and Related Constructions. *Rom. J. Phys.* **2022**, *67*, 101–117.
- Barna, I.F.; Mátyás, L. Advanced Analytic Self-Similar Solutions of Regular and Irregular Diffusion Equations. *Mathematics* **2022**, *10*, 3281. [[CrossRef](#)]
- Mátyás, L.; Barna, I.F. Self-similar and traveling wave solutions of diffusion equations with concentration dependent diffusion coefficients. *Rom. J. Phys.* **2024**, *69*, 106. [[CrossRef](#)]
- Jalghaf, H.K.; Kovács, E.; Majár, J.; Nagy, A.; Askar, A.H. Explicit Stable Finite Difference Methods for Diffusion-Reaction Type Equations. *Mathematics* **2021**, *9*, 3308. [[CrossRef](#)]
- Nagy, A.; Saleh, M.; Omle, I.; Kareem, H.; Kovács, E. New Stable, Explicit, Shifted-Hopscotch Algorithms for the Heat Equation. *Math. Comput. Appl.* **2021**, *26*, 61. [[CrossRef](#)]
- Jalghaf, H.K.; Kovács, E.; Bolló, B. Conduction, Convection, and Bridging. *Buildings* **2022**, *12*, 1365. [[CrossRef](#)]
- Landau, L.; Rumer, G. The Cascade Theory of Electronic Showers. *Proc. R. Soc. Math. Phys. Eng. Sci.* **1938**, *66*, 213. [[CrossRef](#)]
- Smith, R. *Atomic & Ion Collisions in Solids and at Surfaces: Theory, Simulation and Applications*; Cambridge University Press: Cambridge, UK, 1997.
- Cardy, J.; Falkovich, G.; Gawędzki, K. *Non-Equilibrium Statistical Mechanics and Turbulence*; London Mathematical Society Lecture Note Series; Cambridge University Press: Cambridge, UK, 2008; Volume 355.
- Gomperts, B.D.; Tatham, P.E. *Signal Transduction*; Academic Press: Cambridge, MA, USA, 2003.
- Carpenter, R.S.; Kitchell, J.F. *The Trophic Cascade in Lakes*; Cambridge Studies in Ecology; Cambridge University Press: Cambridge, UK, 1996.
- Sirinivasan, S.K. *Stochastic Theory and Cascade Process*; Elsevier: Amsterdam, The Netherlands, 1969.
- Brahim, A.S.; Le Grand, B.; Latapy, M. Diffusion Cascades: Spreading Phenomena in Blog Network Communities. *Parallel Process. Lett.* **2012**, *22*, 1240002. [[CrossRef](#)]
- Liu, X.; Wang, H.; Bouyer, A. A cascade information diffusion prediction model integrating topic features and cross-attention. *J. King Saud Univ. Comput. Inf. Sci.* **2023**, *35*, 101852. [[CrossRef](#)]
- Wit, A.D. Spatial Patterns and Spatiotemporal Dynamics in Chemical Systems. *Adv. Chem. Phys.* **1999**, *109*, 435. [[CrossRef](#)]
- Murray, D. *Mathematical Biology*; Springer: Berlin/Heidelberg, Germany, 2003.
- Kassner, K. *Pattern Formation in Diffusion-Limited Crystal Growth*; World Scientific: Singapore, 1996.

30. Wilhelmsson, H.; Lazzaro, E. *Reaction & Diffusion Problems in the Physics of Hot Plasma*; Institute of Physics Publishing Bristol and Philadelphia: Bristol, UK, 2001.
31. Kremp, D.; Schlages, M.; Bonitz, M.; Bornath, T. Reaction and diffusion in dense nonideal plasmas. *Phys. Fluids* **1993**, *5*, 216. [[CrossRef](#)]
32. Aitani, A.M. Oil Refining and Products. In *Encyclopedia of Energy*; Cleveland, C.J., Ed.; Elsevier: New York, NY, USA, 2004; pp. 715–729. [[CrossRef](#)]
33. Otero, R.; Vázquez de Parga, A.; Gallego, J. Electronic, structural and chemical effects of charge-transfer at organic/inorganic interfaces. *Surf. Sci. Rep.* **2017**, *72*, 105–145. [[CrossRef](#)]
34. Sedov, L.I. *Similarity and Dimensional Methods in Mechanics*; CRC Press: Boca Raton, FL, USA, 1993.
35. Zel'dovich, Y.; Raizer, P.Y. *Physics of Shock Waves and High Temperature Hydrodynamic Phenomena*; Academic Press: Cambridge, MA, USA, 1966.
36. Barenblatt, G.I. *Scaling, Self-Similarity, and Intermediate Asymptotics: Dimensional Analysis and Intermediate Asymptotics*; Cambridge University Press: Cambridge, UK, 1996.
37. Barna, I.F.; Bognar, G.; Hriczo, K. Self-Similar Analytic Solution of the Two-Dimensional Navier-Stokes Equation with a Non-Newtonian Type of Viscosity. *Math. Model. Anal.* **2016**, *21*, 83. [[CrossRef](#)]
38. Olver, F.W.J.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W. (Eds.) *NIST Handbook of Mathematical Functions*; Cambridge University Press: Cambridge, UK, 2010. Available online: <https://dlmf.nist.gov/> (accessed on 1 September 2024).
39. Mátyás, L.; Barna, I. Even and odd self-similar solutions of diffusion equation for infinite horizon. *Universe* **2023**, *9*, 264. [[CrossRef](#)]
40. Potter, M.; O'Malley, A.; Chapman, S.; Kezina, J.; Newland, S.; Silverwood, I.; Mukhopadhyay, S.; Carravetta, M.; Mezza, T.; Parker, S.; et al. Understanding the Role of Molecular Diffusion and Catalytic Selectivity in Liquid-Phase Beckmann Rearrangement. *ACS Catal.* **2017**, *7*, 2926. [[CrossRef](#)]
41. Cross, M.; Hohenberg, P. Pattern formation outside of equilibrium. *Rev. Mod. Phys.* **1993**, *65*, 851. [[CrossRef](#)]
42. Cross, M.; Greenside, H. *Pattern Formation and Dynamics in Nonequilibrium Systems*; Cambridge University Press: Cambridge, UK, 2009.
43. Gross, E.P. Structure of a quantized vortex in boson systems. *Nuovo C* **1961**, *20*, 454. [[CrossRef](#)]
44. Pitaevskii, L.P. Vortex lines in an imperfect Bose gas. *Sov. Phys. JETP* **1961**, *13*, 451.
45. Ronveaux, A. *Heun's Differential Equations*; Oxford University Press: Oxford, UK, 1995.
46. Stoker, H.S. *General, Organic, and Biological Chemistry*; Brooks/Cole: Pacific Grove, CA, USA, 2010.
47. Antonov, L. *Tautomerism: Concepts and Applications in Science and Technology*; Wiley: Hoboken, NJ, USA, 2016.
48. Mátyás, L.; Gaspard, P. Entropy production in diffusion-reaction systems: The reactive random Lorentz gas. *Phys. Rev. E* **2005**, *71*, 036147. [[CrossRef](#)] [[PubMed](#)]
49. Vanag, V.K.; Epstein, I.R. Cross-diffusion and pattern formation in reaction-diffusion systems. *Phys. Chem. Chem. Phys.* **2015**, *11*, 897–912. [[CrossRef](#)] [[PubMed](#)]
50. Hu, Y. Simulation of a partially depleted absorber (PDA) photodetector. *Opt. Express* **2015**, *23*, 20402. [[CrossRef](#)] [[PubMed](#)]
51. McLachlan, N.W. *Theory and Applications of Mathieu Functions*; Dover: Mineola, NY, USA, 1964.

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