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On Killing vectors in initial-value problems for asymptotically flat spacetimes

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Abstract. The existence of symmetries in asymptotically flat spacetimes is studied from the viewpoint of initial-value problems. General necessary and sufficient (implicit) conditions are given for the existence of Killing vector fields in the asymptotic characteristic and in the hyperboloidal initial-value problem (both of which are formulated on the conformally compactified spacetime manifold).

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1. Introduction

The most convenient way of considering the far fields of isolated gravitational systems is to use the conformal technique introduced by Penrose (see in [19]). In this setting one works on the conformally extended spacetime manifold where points at infinity (with respect to the physical metric) are glued to the physical spacetime manifold, i.e. they are represented on this extended, unphysical spacetime manifold by regular points. This means that one works on finite regions of the unphysical spacetime, where one can use all the tools of the standard, local differential geometry to perform calculations, so one avoids the determination of limits at infinity. Of course, not all types of spacetime admit the construction of conformal infinities, those where the conformal extension can be performed are called asymptotically simple. Asymptotically simple spacetimes, where the cosmological constant vanishes, are asymptotically flat. Spacetimes representing isolated gravitational systems are supposed to be asymptotically flat.

Several well defined initial-value problems can be formulated on the extended, unphysical spacetime manifold for asymptotically flat spacetimes, e.g. the following initial-value problems have been studied extensively in the literature.

- In the asymptotic characteristic initial-value problem the data are given on past null infinity $\mathcal{J}^-$ and on an incoming null hypersurface $\mathcal{N}$ which intersects $\mathcal{J}^-$ in a spacelike surface $\mathcal{Z}$ diffeomorphic to $\mathcal{S}^2$ (the problem could be formulated analogously for future null infinity $\mathcal{J}^+$ with an intersecting, outgoing null hypersurface, as well).
- In the hyperboloidal initial-value problem the data are given on a (three-dimensional) spacelike hypersurface $\mathcal{S}$ intersecting future null infinity $\mathcal{J}^+$ in a spacelike surface $\mathcal{Z}$ diffeomorphic to $\mathcal{S}^2$. The term 'hyperboloidal' comes from the fact that the physical metric on $\mathcal{S}$ behaves near $\mathcal{Z}$ like that of a space with constant negative curvature. This

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problem is not time symmetric in the sense that, in contrast to the problem formulated with respect to $\mathcal{J}^+$, the Cauchy development of a hyperboloidal hypersurface intersecting past null infinity $\mathcal{J}^-$ does not extend up to null infinity.

These initial-value problems are solved, i.e. there are uniqueness and existence theorems for these cases (see in [6, 7, 17], a review can be found in [8]).

- The standard Cauchy problem has not yet been completely solved. It is not known what kind of asymptotic conditions need be imposed on the initial data in order to get smooth null infinity in the time evolution, i.e. to get asymptotically flat spacetime in the sense defined by Penrose (the current state of the research is reviewed in [9]).

Working on the unphysical spacetime provides some advantage even for numerical calculations, because the evolution can be calculated over a finite grid covering a conformally compactified initial spacelike hypersurface (some recent results can be found in [3–5, 12–16]).

In the present paper we want to formulate initial-value problems for Killing vector fields in asymptotically flat spacetimes. Our results will be applicable for both of the hyperboloidal and the asymptotic characteristic initial-value problems. We will impose general (implicit) conditions on the initial data, which guarantee the existence of Killing vector fields in the time evolution. Our method is essentially the same as in [20], where the same problem was solved on the physical spacetime manifold for all types of initial-value problems which could be relevant there (see also [10, 21]). However, here we will study the above introduced asymptotic characteristic and hyperboloidal initial-value problems, so we have to work in terms of the conformal quantities on the extended, unphysical spacetime manifold. The scheme of our proof is essentially the same as in [20], however the corresponding calculations are much more complicated. First we will discuss the problem in general, then two examples, spacetimes with Klein–Gordon and Maxwell fields, will be studied in more detail in the appendix.

The symmetry properties of asymptotically flat spacetimes have already been studied by many authors. It is known what kinds of symmetries are admitted by asymptotic flatness and what kinds of additional restrictions are implied by the presence of Killing vector fields. This field already has a huge literature, the investigations are also extended for electrovacuum spacetimes (see [1, 2]) and for spacetimes with non-smooth but polyhomogenous null infinity [18] (these papers also contain a well selected list of the most important previous papers). All these investigations are done by considering asymptotic solutions for the relevant equations (Einstein(–Maxwell) and Killing equations). We do not plan to reproduce the above results, but just to show that analogous investigations can be performed by working exclusively in terms of the conformal quantities. We will consider this topic from the point of view of initial-value problems. With the help of our method one can (in principle) directly connect the initial data and the symmetries of the corresponding solution. This treatment could be useful for numerical investigations.

2. Killing fields on the unphysical spacetime

In the following we use the notation that quantities defined with respect to the physical spacetime manifold $(\tilde{M}, \tilde{g}_{ab})$ have a tilde, and the indices of such (tensorial) quantities will be raised and lowered by the physical metric $(\tilde{g}_{ab}, \tilde{g}^{ab})$.

Let $\tilde{\eta}^a$ be a Killing vector field on the physical spacetime manifold $(\tilde{M}, \tilde{g}_{ab})$, i.e. which satisfies the equation $\mathcal{L}_{\tilde{\eta}}\tilde{g}_{ab} = 0$. From now on, $\mathcal{L}$ will denote the Lie derivative of the corresponding quantity (in the previous case it was taken with respect to the vector field $\tilde{\eta}^a$). The vector field $\tilde{\eta}^a$ has a unique, smooth extension η^a (with $\tilde{\eta}^a = \eta^a|_{\tilde{M}}$) to the conformal spacetime manifold (M, g_{ab}, Ω), where M, Ω and $g_{ab} = \Omega^2\tilde{g}_{ab}$ denote the extended, unphysical spacetime, the conformal factor (which vanishes at null infinity) and the unphysical metric, respectively. The vector field η^a is tangential to null infinity $\mathcal{J}$ (in this paper the symbol $\mathcal{J}$ will denote both future and past null infinity, i.e. $\mathcal{J}^+$ and $\mathcal{J}^-$, respectively), i.e. the condition $\eta(\Omega)|_{\mathcal{J}} = 0$ is satisfied [11]. On (M, g_{ab}, Ω) the vector field η^a is a conformal Killing vector field, i.e. the equation

$$\mathcal{L}_\eta g_{ab} = \nabla_a\eta_b + \nabla_b\eta_a = \eta(\omega)\, g_{ab} \qquad \text{with} \quad \omega = \ln(\Omega^2) \tag{2.1}$$

is satisfied†. Here ∇_a denotes the Levi-Civita differential operator corresponding to the conformal metric g_{ab}. Substituting equation (2.1) into the definition $\nabla_a\nabla_b\eta_c - \nabla_b\nabla_a\eta_c = R_{abc}{}^f\eta_f$ of the curvature tensor, and then contracting it with the unphysical metric g^{ab}, we find the equation

$$\Box\eta_a + \nabla_a\eta(\omega) + R_a{}^f\eta_f = 0, \tag{2.2}$$

where we have introduced the D'Alambert operator $\Box := \nabla_f\nabla^f$. The above equation is satisfied by all conformal Killing vector fields. Moreover, equation (2.2) is a linear wave equation. We can formulate, by prescribing suitable initial conditions, well defined initial-value problems for an arbitrary vector field η_a that satisfies the above equation. We will show in the following that vector fields satisfying equation (2.2), supplied additionally with appropriate initial data, do satisfy equation (2.1) on the unphysical spacetime. This means they are proper Killing vector fields in the physical spacetime.

We obtain by some algebra after differentiating equation (2.2) the equation

$$\begin{aligned}0 = \Box\nabla_a\eta_b &+ (\nabla_a R_b{}^f - \nabla_b R_a{}^f + \nabla^f R_{ab})\eta_f + 2\, R_a{}^e{}_b{}^f\nabla_e\eta_f - R_a{}^f\nabla_f\eta_b \\ &+ R_b{}^f\nabla_a\eta_f + \nabla_a\nabla_b\eta(\omega).\end{aligned} \tag{2.3}$$

Introducing the tensor field

$$C_{ab} = \nabla_a\eta_b + \nabla_b\eta_a - \eta(\omega)g_{ab}, \tag{2.4}$$

equation (2.3) implies the expression

$$0 = \Box C_{ab} + 2\, R_a{}^e{}_b{}^f C_{ef} - R_a{}^f C_{fb} - R_b{}^f C_{fa} + g_{ab}\Box\eta(\omega) + 2\,\mathcal{L}_\eta R_{ab} + 2\,\nabla_a\nabla_b\eta(\omega). \tag{2.5}$$

The last term here can be rewritten as

$$\begin{aligned}2\,\nabla_a\nabla_b\eta(\omega) = {}& (\nabla_a C_{bf} + \nabla_b C_{af} - \nabla_f C_{ab})\nabla^f\omega + \mathcal{L}_\eta(\nabla_a\omega\nabla_b\omega + 2\,\nabla_a\nabla_b\omega) \\ & - g_{ab}(\nabla^f\omega)\mathcal{L}_\eta\nabla_f\omega,\end{aligned} \tag{2.6}$$

while for $\Box\eta(\omega)$ we obtain

$$\Box\eta(\omega) = C_{ab}\nabla^a\nabla^b\omega - \tfrac{1}{2}C_{ab}(\nabla^a\omega)\nabla^b\omega + \tfrac{1}{3}\eta(\omega)(\tilde{R} - R) + \tfrac{1}{3}\mathcal{L}_\eta(\tilde{R} - R), \tag{2.7}$$

where we have introduced $\tilde{R} = g^{ef}\tilde{R}_{ef}$, while R denotes the curvature scalar of the unphysical spacetime. Deriving the above equations, we repeatedly exploited the fact that η_a is a solution of the wave equation (2.2). Substituting the formulae (2.6) and (2.7) into (2.5) and utilizing that

$$2\,\mathcal{L}_\eta(R_{ab} + \nabla_a\nabla_b\omega) = \mathcal{L}_\eta\big\{2\,\tilde{R}_{ab} - \tfrac{1}{3}g_{ab}(\tilde{R} - R) + \tfrac{1}{2}g_{ab}(\nabla_f\omega)\nabla^f\omega - (\nabla_a\omega)\nabla_b\omega\big\}, \tag{2.8}$$

where we have used the conformal transformation formula

$$R_{ab} = \tilde{R}_{ab} + \frac{3}{\Omega^2}g_{ab}(\nabla_f\Omega)\nabla^f\Omega - \frac{1}{\Omega}\big\{2\,\nabla_a\nabla_b\Omega + g_{ab}\nabla_f\nabla^f\Omega\big\} \tag{2.9}$$

† It is worth emphasizing that $\eta(\omega)$ is a regular expression even on null infinity, where Ω vanishes.

for the Ricci tensor, we arrive at our evolution equation

$$
\begin{aligned}
0 = \Box C_{ab} + 2\, R_a{}^e{}_b{}^f C_{ef} - R_a{}^f C_{fb} - R_b{}^f C_{fa} + \left\{\nabla_a C_{bf} + \nabla_b C_{af} - \nabla_f C_{ab}\right\}\nabla^f \omega \\
+\left\{\tfrac{1}{2}(\nabla_f \omega)\nabla^f \omega - \tfrac{1}{3}(\tilde{R} - R)\right\} C_{ab} \\
-g_{ab}\left\{(\nabla^e \omega)\nabla^f \omega - \nabla^e \nabla^f \omega\right\} C_{ef} + 2\mathcal{L}_\eta \tilde{R}_{ab}
\end{aligned}
\tag{2.10}
$$

for the tensor field C_{ab}.

3. Vacuum spacetimes

Equation (2.10) is a second-order, linear, hyperbolic partial differential equation for the tensor field C_{ab}, which is additionally homogeneous in vacuum ($\tilde{R}_{ab} = 0$). The following assertion is thus a simple consequence of the general existence and uniqueness theorems for wave equations (cf [22]).

Theorem 3.1. *Let* (M, g_{ab}, Ω) *denote some conformally compactified, asymptotically flat vacuum spacetime. If the vector field* η_a *is a non-trivial solution of the evolution equation (2.2); furthermore, if the tensor field* C_{ab} *vanishes on the initial surfaces (hypersurface) of the considered asymptotic characteristic (hyperboloidal) initial-value problem, then* $\tilde{\eta}^a = \eta^a|_{\tilde{M}}$ *is a Killing vector field on the considered region of the physical spacetime.*

The regularity of the principal part of (2.10) allows the use of the standard energy estimate methods for proving the uniqueness of the $\{C_{ab} \equiv 0\}$ (i.e. η^a is a conformal Killing vector) solution.

Now we turn to the more general case where matter fields are also present in the spacetime. First, we will derive some general results, then finally we discuss the cases of a massless scalar and an electromagnetic field.

4. Spacetimes with matter fields

We start with some general assumption on the matter fields admitted in spacetimes which will be discussed in our following studies. We will suppose that the energy-impulse tensor has the structure

$$
\tilde{T}_{ab} = \tilde{T}_{ab}(\tilde{\Phi}^{(i)}_A, \tilde{\nabla}_e \tilde{\Phi}^{(i)}_A, \tilde{g}_{ef}),
\tag{4.1}
$$

i.e. it depends on some matter fields $\tilde{\Phi}^{(i)}_A \equiv \tilde{\Phi}^{(i)}_{a_1 \ldots a_n}$ (where capital indices such as '$A, B, \ldots$' are multi-indices denoting a collection '$a_1 a_2 \ldots$' of covariant indices, while 'i' is just to label several matter fields), on their first covariant derivatives and on the physical metric. The fields $\tilde{\Phi}^{(i)}_A$ are supposed to have regular limits at null infinity $\mathcal{J}$, so they can be smoothly extended to well defined tensor fields $\Phi^{(i)}_A$ on the unphysical spacetime where $\tilde{\Phi}^{(i)}_A = \Phi^{(i)}_A|_{\tilde{M}}$ is satisfied. Assuming the Einstein equation holds, we obtain an expression similar to (4.1) for the physical Ricci tensor†

$$
\tilde{R}_{ab} = \tilde{R}_{ab}(\tilde{\Phi}^{(i)}_A, \tilde{\nabla}_e \tilde{\Phi}^{(i)}_A, \tilde{g}_{ef}).
\tag{4.2}
$$

Equation (2.10) contains the Lie derivative of the physical Ricci tensor

$$
\mathcal{L}_\eta \tilde{R}_{ab} = \sum_i \frac{\partial \tilde{R}_{ab}}{\partial \tilde{\Phi}^{(i)}_A} \mathcal{L}_\eta \tilde{\Phi}^{(i)}_A + \sum_i \frac{\partial \tilde{R}_{ab}}{\partial \tilde{\nabla}_e \tilde{\Phi}^{(i)}_A} \mathcal{L}_\eta \tilde{\nabla}_e \tilde{\Phi}^{(i)}_A + \frac{\partial \tilde{R}_{ab}}{\partial \tilde{g}_{ef}} \mathcal{L}_\eta \tilde{g}_{ef}.
\tag{4.3}
$$

† It is worth noting that we could have started, just like in [20], by imposing the conditions (4.2) and (4.10); the analysis itself is independent of the exact form of the Einstein equation.

Like the matter fields $\tilde{\Phi}_A^{(i)}$, their Lie derivatives $\mathcal{L}_\eta \tilde{\Phi}_A^{(i)}$ are also well defined tensor fields on the whole unphysical spacetime, more precisely $\mathcal{L}_\eta \tilde{\Phi}_A^{(i)} = \mathcal{L}_\eta \Phi_A^{(i)}|_{\tilde{M}}$ is satisfied. Therefore, we can also omit the tildes from $\Phi_A^{(i)}$, and the Lie derivative appearing in the second term of equation (4.3) can be rewritten as

$$\mathcal{L}_\eta(\tilde{\nabla}_e \Phi^{(i)}_{a_1\ldots a_n}) = \tilde{\nabla}_e \mathcal{L}_\eta \Phi^{(i)}_{a_1\ldots a_n} - \sum_{j=1}^{n} (\tilde{\nabla}\mathcal{L}_\eta \tilde{g})_{ea_j f}\, \tilde{g}^{fh} \Phi^{(i)}_{a_1\ldots h\ldots a_n}, \tag{4.4}$$

where we have used the abbreviation

$$(\tilde{\nabla}\mathcal{L}_\eta \tilde{g})_{ea_j f} = \tfrac{1}{2}\big(\tilde{\nabla}_e \mathcal{L}_\eta \tilde{g}_{a_j f} + \tilde{\nabla}_{a_j} \mathcal{L}_\eta \tilde{g}_{ef} - \tilde{\nabla}_f \mathcal{L}_\eta \tilde{g}_{ea_j}\big). \tag{4.5}$$

Both of the above equations contain the Levi-Civita differential operator $\tilde{\nabla}$ induced by the physical metric $\tilde{g}_{ab}$. In order to extend these expressions into the unphysical spacetime first we have to change to the operators ∇ induced by the conformal metric g_{ab}, i.e. we have to apply the conformal transformations

$$\begin{aligned} \tilde{\nabla}_e \mathcal{L}_\eta \Phi^{(i)}_{a_1\ldots a_n} &= \nabla_e \mathcal{L}_\eta \Phi^{(i)}_{a_1\ldots a_n} + \sum_{j=1}^{n} \hat{\Gamma}_{ea_j}{}^{f} \mathcal{L}_\eta \Phi^{(i)}_{a_1\ldots f\ldots a_n}, \\ \tilde{\nabla}_e \mathcal{L}_\eta \tilde{g}_{ab} &= \frac{1}{\Omega^2}\nabla_e C_{ab} + \frac{1}{\Omega^2}\left(\hat{\Gamma}_{ea}{}^{f} C_{fb} + \hat{\Gamma}_{eb}{}^{f} C_{af} - \frac{2}{\Omega} C_{ab} \nabla_e \Omega\right), \end{aligned} \tag{4.6}$$

where we have used the symbols

$$\hat{\Gamma}_{ab}{}^{c} = \frac{2}{\Omega}\left(\delta^c_{(a}\nabla_{b)}\Omega - \tfrac{1}{2} g_{ab} g^{cd} \nabla_d \Omega\right). \tag{4.7}$$

We have also applied the relation

$$\mathcal{L}_\eta \tilde{g}_{ab} = \frac{C_{ab}}{\Omega^2} \tag{4.8}$$

which follows directly from the definition (2.4) of the tensor field C_{ab}. Substituting all of the above-quoted formulae into expression (4.3) we obtain an equation with the structure

$$\mathcal{L}_\eta \tilde{R}_{ab} = \mathcal{A}_{ab}(\mathcal{L}_\eta \Phi_A^{(i)}) + \mathcal{B}_{ab}(\nabla_e \mathcal{L}_\eta \Phi_A^{(i)}) + \mathcal{C}_{ab}(C_{ef}) + \mathcal{D}_{ab}(\nabla_e C_{fg}), \tag{4.9}$$

where all of $\mathcal{A}_{ab}$, $\mathcal{B}_{ab}$, $\mathcal{C}_{ab}$ and $\mathcal{D}_{ab}$ are linear, homogeneous functions in their indicated arguments. However, as we will see later in explicit examples, some of these functions may also contain terms with negative powers of the conformal factor Ω which vanishes on null infinity. This means that singular terms can also appear on the right-hand side of (4.9).

Let us suppose that the matter fields $\tilde{\Phi}_A^{(i)} = \Phi_A^{(i)}|_{\tilde{M}}$ satisfy the field equations

$$\tilde{\nabla}_a \tilde{\nabla}^a \tilde{\Phi}_A^{(i)} = F_A^{(i)}(\tilde{\Phi}_B^{(j)}, \tilde{\nabla}_e \tilde{\Phi}_B^{(j)}, \tilde{g}_{ef}), \tag{4.10}$$

where $F_A^{(i)}$ denote some smooth functions of the indicated arguments (the matter fields are admitted to be coupled to each other). Calculating the Lie derivative of the previous equation, and then performing the same transformations as above, we can derive the evolution equations

$$\Box \mathcal{L}_\eta \Phi_A^{(i)} = \mathcal{E}_A^{(i)}(\mathcal{L}_\eta \Phi_B^{(j)}) + \mathcal{F}_A^{(i)}(\nabla_e \mathcal{L}_\eta \Phi_B^{(j)}) + \mathcal{G}_A^{(i)}(C_{ef}) + \mathcal{H}_A^{(i)}(\nabla_e C_{fg}) \tag{4.11}$$

for the Lie derivatives of the matter fields (we have already performed the conformal transformations (4.6)–(4.8), as well). The functions $\mathcal{E}_A^{(i)}$, $\mathcal{F}_A^{(i)}$, $\mathcal{G}_A^{(i)}$ and $\mathcal{H}_A^{(i)}$ are linear and homogeneous in their indicated arguments. However, their regularity, like above at (4.9), depends on the concrete physical model which is investigated.

Equations (2.10) (in view of equation (4.9)) and (4.11) are composed of a system of linear, homogeneous wave equations for the variables C_{ab} and $\mathcal{L}_\eta \Phi_A^{(i)}$. The right-hand sides can contain terms which are singular at null infinity; however, the principal parts are always regular. This allows us to apply the general theorems (see in [22]) to prove the uniqueness of the $\{C_{ab} \equiv 0, \mathcal{L}_\eta \Phi_A^{(i)} \equiv 0\}$ solution. This means that the following assertion is just a simple consequence of the general theorems.

Theorem 4.1. *Let (M, g_{ab}, Ω) denote some conformally compactified asymptotically flat spacetime containing some matter fields $\Phi_A^{(i)}$ which satisfy the evolution equations (4.10). If the vector field η_a is a non-trivial solution of the wave equation (2.2); furthermore, if $\mathcal{L}_\eta \Phi_A^{(i)}$ and the tensor field C_{ab} vanish on the initial surfaces (hypersurface) of the considered asymptotic characteristic (hyperboloidal) initial-value problem, then $\tilde{\eta}^a = \eta^a|_{\tilde{M}}$ is a Killing vector field on the considered region of the physical spacetime.*

In the appendix we will consider two examples, the massless scalar and the electromagnetic field, in more detail. First, it is useful to write down the actual formulae where one can observe explicitly the nature of the singularities appearing on $\mathcal{J}$, caused by the $\frac{1}{\Omega}$ terms. Secondly, the field equations for the electromagnetic field are not automatically of the form (4.10), one has to impose suitable gauge conditions in order to apply the above general results.

5. Summary

We have derived general necessary and sufficient initial conditions for Killing vector fields in the asymptotic characteristic and hyperboloidal initial-value problems. These conditions are implicit in the sense that they have to be evaluated for the components of the vector field η^a. The existence of Killing vector fields depends on the existence of $\eta^a \neq 0$ solutions for the conditions derived above for the initial data (equation (2.2), $C_{ab} = 0$ and $\mathcal{L}_\eta \Phi_A^{(i)} = 0$ have to be evaluated on the initial surfaces (hypersurface)). Unfortunately, the evaluation of these general implicit conditions cannot be done in such a general way as we treated the whole problem in this paper. For instance, the calculations are fundamentally different for the asymptotic characteristic and for the hyperboloidal initial-value problems. The case of the asymptotic characteristic initial-value problem is the simpler one, because the initial surfaces are characteristics for equation (2.2), so they induce propagation equations for the transversal derivatives of the Killing field. A further important distinction arises from the relationship of the Killing vector field to the initial surfaces, i.e. whether it is tangential or transversal (on null infinity it has to be transversal). Unfortunately, there are a lot of possible configurations which should be analysed separately. Here we presented just the framework and the general results, future work is planned for studying some applications.

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Appendix. Massless scalar and electromagnetic field

Let as consider a massless scalar field $\tilde{\Phi}$ on the physical spacetime manifold $(\tilde{M}, \tilde{g}_{ab})$ with the evolution equation

$$\tilde{\nabla}_a \tilde{\nabla}^a \tilde{\Phi} = 0, \tag{A.1}$$

and with energy-impulse tensor

$$\tilde{T}_{ab} = \tilde{\nabla}_a\tilde{\Phi}\tilde{\nabla}_b\tilde{\Phi} - \tfrac{1}{2}\tilde{g}_{ab}\tilde{\nabla}_f\tilde{\Phi}\tilde{\nabla}^f\tilde{\Phi}. \tag{A.2}$$

The scalar field $\tilde{\Phi}$ is supposed to have a regular limit at null infinity, so it has a unique extension Φ (with $\tilde{\Phi} = \Phi|_{\tilde{M}}$) which is regular on the whole unphysical spacetime (M, g_{ab}, Ω). By the Einstein equations the Lie derivative of the physical Ricci tensor with respect to η^a is simply

$$\mathcal{L}_\eta\tilde{R}_{ab} = 8\pi\left[(\nabla_a\mathcal{L}_\eta\Phi)\nabla_b\Phi + (\nabla_a\Phi)\nabla_b\mathcal{L}_\eta\Phi\right]. \tag{A.3}$$

Here we already performed the conformal transformation, so the above expression is already written in terms of the unphysical quantities. We can recognize that (A.3) is a completely regular expression on the whole conformal spacetime manifold.

We can evaluate (4.11), as well. After some longer but straightforward calculations we arrive at the equation

$$\begin{aligned} 0 = {} & \Box\mathcal{L}_\eta\Phi - (\nabla^e\nabla^f\Phi)C_{ef} - g^{ef}\nabla^h\Phi\left\{\nabla_e C_{fg} - \tfrac{1}{2}\nabla_g C_{ef}\right\} \\ & -2\frac{\nabla^e\Omega}{\Omega}\left\{\nabla_e\mathcal{L}_\eta\Phi - (\nabla^f\Phi)C_{ef}\right\}. \end{aligned} \tag{A.4}$$

Equations (2.10) (in view of (A.3)) and (A.4) constitute a linear, homogeneous system of wave equations for the variables C_{ab} and $\mathcal{L}_\eta\Phi$. The following statement is thus a special case of the general theorem formulated in the previous section.

Proposition A.1. *Let* (M, g_{ab}, Ω) *denote some conformally compactified asymptotically flat spacetime containing some massless scalar field in the considered region. If the vector field* η_a *is a non-trivial solution of the evolution equation (2.2), furthermore if* $\mathcal{L}_\eta\Phi$ *and the tensor field* C_{ab} *vanish on the initial surfaces (hypersurface) of the considered asymptotic characteristic (hyperboloidal) initial-value problem, then* $\tilde{\eta}^a = \eta^a|_{\tilde{M}}$ *is a Killing vector field on the considered region of the physical spacetime.*

Now we turn to the study of the electromagnetic field. The field equations in the physical spacetime are given by

$$({}^*\mathrm{d}\tilde{F})_a = 0, \qquad \tilde{F}_{ab} = (\mathrm{d}\tilde{A})_{ab}, \tag{A.5}$$

where d denotes the exterior differential of the corresponding quantity and the star indicates the Hodge-dual. The vector potential is considered as the restriction to $\tilde{M}$ of a $\tilde{A}_a = A_a|_{\tilde{M}}$ 1-form field A_a which is regular on the whole conformally extended spacetime manifold. The electromagnetic field equations are conformally invariant, so the previous equations can be rewritten as

$$\nabla^a F_{ab} = 0, \qquad F_{ab} = \nabla_a A_b - \nabla_b A_a, \tag{A.6}$$

where we already used the conformal Levi-Civita differential operator induced by the unphysical metric g_{ab}. The energy-impulse tensor and the physical Ricci tensor can be written, using the Einstein equations, as simply

$$\begin{aligned} \tilde{T}_{ab} &= F_{ae}F_{bf}\tilde{g}^{ef} - \tfrac{1}{4}\tilde{g}_{ab}F_{ce}F_{df}\tilde{g}^{cd}\tilde{g}^{ef}, \\ \tilde{R}_{ab} &= 8\pi\left[F_{ae}F_{bf}\tilde{g}^{ef} - \tfrac{1}{4}\tilde{g}_{ab}F_{ce}F_{df}\tilde{g}^{cd}\tilde{g}^{ef}\right], \end{aligned} \tag{A.7}$$

respectively. It is easy to check that the Lie derivative (4.9) of the physical Ricci tensor now takes the form

$$\mathcal{L}_\eta\tilde{R}_{ab} = \Omega^2\left\{\mathcal{A}_{ab}(\mathcal{L}_\eta A_e) + \mathcal{C}_{ab}(C_{ef})\right\}, \tag{A.8}$$

where $\mathcal{A}_{ab}$ and $\mathcal{C}_{ab}$ are some regular functions, homogeneous and isotropic in their indicated arguments.

The evaluation of (4.11) requires a bit more work. Calculating the Lie derivative of the first equation from (A.6) after some lengthy but straightforward calculation we arrive at

$$\begin{aligned}0 = \Box\mathcal{L}_\eta A_a - g^{fg}\{\nabla^e\nabla_f A_a - \nabla^e\nabla_a A_f\}C_{eg} + g^{gh}\nabla^f A_h\{\nabla_g C_{af} - \nabla_f C_{ag}\} - R_a{}^g\mathcal{L}_\eta A_g \\ -\nabla_a\{(\mathcal{L}_\eta + \eta(\omega))(\nabla^f A_f - A(\omega)) + g^{ef}g^{gh}(\nabla_e A_g - A_e\nabla_g\omega)C_{fh} \\ +(\nabla^f\omega)\mathcal{L}_\eta A_f\}.\end{aligned} \tag{A.9}$$

During the derivation of the previous equation we used only identity (4.4), equation (4.8) and the evolution equation (2.2) satisfied by the vector field η_a.

It is easy to check that in terms of the conformal quantities the Lorentz gauge condition can be rewritten as

$$\tilde{\nabla}^f\tilde{A}_f = \Omega^2[\nabla^f A_f - A(\omega)] = 0. \tag{A.10}$$

This means that in the Lorenz gauge equation (A.9) takes a simpler form

$$\begin{aligned}0 = \Box\mathcal{L}_\eta A_a - g^{fg}\{\nabla^e\nabla_f A_a - \nabla^e\nabla_a A_f\}C_{eg} - g^{gh}\nabla^f A_h\{\nabla_a C_{fg} + \nabla_f C_{ag} - \nabla_g C_{af}\} \\ -R_a{}^g\mathcal{L}_\eta A_g - g^{fh}g^{gk}\{\nabla_a\nabla_h A_k - \nabla_a A_h\nabla_k\omega - A_h\nabla_a\nabla_k\omega\}C_{fg} \\ -(\nabla_a\nabla^f\omega)\mathcal{L}_\eta A_f - (\nabla^f\omega)\nabla_a\mathcal{L}_\eta A_f.\end{aligned} \tag{A.11}$$

This expression already has the structure of (4.11). At this point we can argue like above, i.e. equations (A.11) and (2.10) constitute a linear, homogeneous system of wave equations with a unique $\mathcal{L}_\eta A_a \equiv 0$ and $C_{ab} \equiv 0$ solution. So the following statement is just a special case of the general theorem of the previous section.

Proposition A.2. *Let* (M, g_{ab}, Ω) *denote some conformally compactified asymptotically flat spacetime containing an electromagnetic field in the considered region. Let the vector potential* $\tilde{A}_a$ *be given in the Lorentz-gauge, i.e.* $\tilde{\nabla}^a\tilde{A}_a = 0$. *If the vector field* η_a *is a non-trivial solution of the evolution equation (2.2); furthermore, if* $\mathcal{L}_\eta A_a$ *and the tensor field* C_{ab} *vanish on the initial surfaces (hypersurface) of the considered asymptotic characteristic (hyperboloidal) initial-value problem, then* $\tilde{\eta}_a = \eta_a|_{\tilde{M}}$ *is a Killing vector field in the considered region of the physical spacetime.*

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