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LETTER TO THE EDITOR

A note on the existence of conjugate points

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Abstract. It is shown that a particular integral condition imposed on one of the diagonal components of the tidal-force tensor is sufficient to assure the existence of conjugate points along causal geodesics.

Among the conditions of singularity theorems those assuring the existence of conjugate points play a crucial role. In the early singularity theorems the strong energy condition was used (see [1, 2]). This condition is, in fact, a local restriction on the trace $R_{ab}T^aT^b$ of the tidal-force tensor and might even be violated in some physically interesting cases. For example, it is satisfied neither for some massive scalar fields nor in the inflationary cosmological model (see [4]). Instead of the strong energy condition, Tipler [2], Roman [3] and Borde [4] have set up other global integral conditions implying invariably the whole trace $R_{ab}T^aT^b$. In this letter we shall give, by generalizing Borde's results, a sufficient condition for the existence of conjugate points in terms of integrals of certain components of the tidal-force tensor. The tidal-force tensor is a physical, easy-to-measure quantity, a genuine appearance of gravitation.

Our condition is given in terms of an integral of the tidal force tensor component belonging to some direction orthogonal to the tangent of the geodesic. So it could be useful in studying the anisotropic focusing, for example in the gravitational lenses.

To get our theorem we first recall some results about the behaviour of congruences of causal geodesics. With the usual choice of reference frame the expansion of the causal congruence satisfies the following equation (see [1])

$$\frac{\mathrm{d}\theta_{ij}}{\mathrm{d}t}=-R_{i4j4}-\sum_{k=1}^{n}\omega_{ik}\omega_{kj}-\sum_{k=1}^{n}\theta_{ik}\theta_{kj} \tag{1}$$

where t is an affine parameter (chosen to be the proper time in the timelike case). The notation will be the same as that of [1] except that $n=3$ and $n=2$ throughout the present letter for the timelike case (null case, respectively) and the subscripts i, j and k take the values 1, 2, 3 (1, 2 respectively).

It will be supposed that there exists such a parameter value t_0 that the vorticity tensor ω_{ij} of the congruence vanishes at $\gamma(t_0)$. Consequently, ω_{ij} vanishes identically along the examined geodesic.

One can rewrite the diagonal equations from (1) into the following form

$$\frac{\mathrm{d}\theta_{ii}}{\mathrm{d}t}=-R_{i4i4}-\theta_{ii}^2-\sum_{k=1,k\neq i}^{n}\theta_{ik}^2. \tag{2}$$

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These equations are similar to the Raychaudhuri equation, studied earlier by Borde and others, and have the same properties. Hence the proofs given by Borde in [4] can be generalized easily. We have only to slightly modify his notation and clarify some new possibilities. Before presenting our theorem we prove some lemmas we shall need.

Lemma 1. Let $\gamma:[t_0,+\infty)\to M$ a causal geodesic. Suppose that there is a unit vector E_1 orthogonal to and parallelly propagated along γ so that for any $\varepsilon>0$ there is such a $b>0$ that if $t'>t_0$ there is an interval $I>t'$ of length $\geqslant b$ for which

$$\int_{t_0}^{t} R_{1414}\,\mathrm{d}t \geqslant -\varepsilon \qquad \forall t\in I. \tag{3}$$

Then for any congruence containing γ with $\theta_{11}(t_0)\leqslant 0$ there exists a point $\gamma(\tau)$ such that $t_0<\tau$ and

$$\lim_{t\to\tau}\theta_{11}(t)=-\infty \qquad t<\tau \tag{4}$$

supposing that $\theta_{11}(t)$ is not identically zero along γ.

Proof. The equation from (2) belonging to E_1 takes the form

$$\frac{\mathrm{d}\theta_{11}}{\mathrm{d}t}=-R_{1414}-\theta_{11}^2-\sum_{k=2}^{n}\theta_{1k}^2. \tag{5}$$

Then replace the terms $n^{-1/2}\theta$, $R_{ab}T^aT^b$ and $2\sigma^2$ by θ_{11}, R_{1414} and $\Sigma_{k=2}^{n}\theta_{1k}^2$ respectively in the proof of theorem 1 of [4]. □

Lemma 2. Let $\theta_{11}(t)$ be a solution of equation (5) and $\tilde\theta_{11}(t)$ a solution of equation (5) with the last term on the right-hand side omitted. Then $\tilde\theta_{11}(t_0)=\theta_{11}(t_0)$ implies that

$$\tilde\theta_{11}(t)\leqslant\theta_{11}(t) \qquad t\leqslant t_0 \tag{6a}$$

$$\tilde\theta_{11}(t)\geqslant\theta_{11}(t) \qquad t\geqslant t_0. \tag{6b}$$

Proof. One can prove this lemma simply by making the replacements $\tilde\theta_{11}\to n^{-1/2}\theta_1$, $\theta_{11}\to n^{-1/2}\theta_2$, $R_{1414}\to R_{ab}T^aT^b$ and $\Sigma_{k=2}^{n}\theta_{1k}^2\to 2\sigma^2$ in the proof of lemma 2 of [4]. □

Lemma 3. Let $\gamma(t)$ be a complete causal geodesic. Suppose that there is such a unit vector E_1 orthogonal to and parallelly propagated along γ that for any $\varepsilon>0$ there is such a $b>0$ that for any $t_1<t_2$ there is a pair of intervals I, J of lengths $\geqslant b$ that $I<t_1<t_2<J$ and

$$\int_{t'}^{t''} R_{1414}\,\mathrm{d}t\geqslant-\varepsilon \qquad \forall t'\in I, \forall t''\in J. \tag{7}$$

Equation (5) has then such a solution along γ that there is a pair of parameter values τ_1, τ_2 at which

$$\lim_{t\to\tau_1}\theta_{11}=+\infty \qquad t>\tau_1 \tag{8a}$$

$$\lim_{t\to\tau_2}\theta_{11}=-\infty \qquad t<\tau_2 \tag{8b}$$

supposing that one of the functions $R_{14i4}(t)$ is not identically zero along γ.

Proof. (*a*) Suppose that there is such a parameter value t_0 that $R_{1414}(t_0) \neq 0$. Then the proof can be got via a sensible modification of the proof of theorem 2 of [4].

(*b*) Suppose now that $R_{1414}(t)$ vanishes identically along γ. Then θ_{11} satisfies the equation

$$\frac{\mathrm{d}\theta_{11}}{\mathrm{d}t} = -\theta_{11}^2 - \sum_{k=2}^{n} \theta_{1k}^2. \tag{9}$$

Solving the 'reduced' equation

$$\frac{\mathrm{d}\tilde{\theta}_{11}}{\mathrm{d}t} = -\tilde{\theta}_{11}^2 \tag{10}$$

obtained from (9), we get the following solution

$$\tilde{\theta}_{11}(t) = \left((t-t_0) + \frac{1}{\tilde{\theta}_{11}(t_0)}\right)^{-1}. \tag{11}$$

Let the parameter value t_0 be chosen so that one of the functions $R_{14i4}(t)$ is not identically zero on the interval $[t_0, -\infty)$. Let $\theta_{11}(t)$ be the solution of (9) having the same initial value at t_0 as $\tilde{\theta}_{11}(t)$ has. Then we get by lemma 2 that

$$\theta_{11}(t) \leqslant \left((t-t_0) + \frac{1}{\theta_{11}(t_0)}\right)^{-1} \qquad t \geqslant t_0 \tag{12a}$$

$$\theta_{11}(t) \geqslant \left((t-t_0) + \frac{1}{\theta_{11}(t_0)}\right)^{-1} \qquad t \leqslant t_0. \tag{12b}$$

We can see from (12*a*) that whenever $\theta_{11}(t_0) < 0$ there exists such a finite parameter value $t_2 \leqslant t_0 - 1/\theta_{11}(t_0)$ that $t_0 < t_2$ and

$$\lim_{t \to t_2} \theta_{11}(t) = -\infty \qquad t < t_2. \tag{13}$$

If $\theta_{11}(t_0) = 0$ there will be such a finite parameter value $t_1 > t_0$ that $\theta_{11}(t_1) < 0$. The existence of such a t_1 is a consequence of the conditions of this lemma and the choice of t_0. One can see from equation (1) that one of the functions $\theta_{1k}(t)$ will not be identically zero on the interval $[t_0, +\infty)$, so one can get from (9) that $\theta_{11}(t)$ must become negative somewhere beyond t_0. For this negative initial value $\theta_{11}(t_1)$, replacing t_0 by t_1, we can use the above method and we get such a parameter value t_2 that property (13) is satisfied. Consider now all the solutions of equation (9) with initial values $\theta_{11}(t_0) \leqslant 0$. Since the solutions of a first-order ordinary differential equation cannot change their rank, the parameter value t_2 belonging to the solution with initial value $\theta_{11}(t_0) = 0$ is greater than those belonging to the other solutions. Denote this greatest value by t_2^*.

Now we are going to construct the required solution $\theta_{11}(t)$. Take the parameter value t_2^* and consider a solution with the initial condition $\theta_{11}(t_2^*) < 0$. By repeating the above argument one finds that there exists such a parameter value $\tau_2 > t_2^*$ that

$$\lim_{t \to \tau_2} \theta_{11}(t) = -\infty \qquad t < \tau_2. \tag{14}$$

This solution is either continuous in the interval $[t_0, t_2^*]$ or not. If not then let τ_1 be the greatest parameter value $\tilde{t}$ for which

$$\lim_{t \to \tilde{t}} \theta_{11}(t) = +\infty \qquad \tilde{t} < t_2^*. \tag{15}$$

Whenever the solution in question is continuous $\theta_{11}(t_0)$ must be greater than zero because otherwise it would diverge somewhere before or at t_2^*. Then we have by (12b) for the parameter value τ_1 ($t_0 > \tau_1 > t_0 - 1/\theta_{11}(t_0)$) that

$$\lim_{t \to \tau_1} \theta_{11}(t) = +\infty \qquad t > \tau_1. \tag{16}$$

Consequently, the solution belonging to the initial value $\theta_{11}(t_2^*) < 0$ is the required one. □

Now we prove that the conditions of the above lemma imply not only the existence of a solution for equation (9) obeying the conditions (8a) and (8b) but the existence of a conjugate point pair along γ.

Theorem. Let γ be a complete causal geodesic with affine parameter t and suppose that there is such a unit vector E_1 orthogonal to and propagated parallel to γ that for any $\varepsilon > 0$ there is a $b > 0$ that for any $t_1 < t_2$ there is such a pair of intervals I, J of length $\geqslant b$ that $I < t_1 < t_2 < J$ and

$$\int_{t'}^{t''} R_{1414}\,\mathrm{d}t \geqslant -\varepsilon \qquad \forall t' \in I, \forall t'' \in J. \tag{17}$$

Then, if one of the functions $R_{i4i4}(t)$ is not identically zero there is a pair of conjugate points along γ.

Proof. We shall show that the Raychaudhuri equation has such a solution $\theta(t) = \Sigma_{i=1}^n \theta_{ii}(t)$ that there exists a pair of parameter values τ_1, τ_2 having the following properties: $\theta(t) = \Sigma_{i=1}^n \theta_{ii}(t)$ is continuous in the interval (τ_1, τ_2) and

$$\lim_{t \to \tau_1} \theta(t) = +\infty \qquad t > \tau_1 \tag{18a}$$

$$\lim_{t \to \tau_2} \theta(t) = -\infty \qquad t < \tau_2 \tag{18b}$$

are satisfied. Because of lemma 3 there is such a pair of parameter values $\tau_1' < \tau_2'$ that θ_{11} is continuous in (τ_1', τ_2') and

$$\lim_{t \to \tau_1'} \theta_{11}(t) = +\infty \qquad t > \tau_1' \tag{19a}$$

$$\lim_{t \to \tau_2'} \theta_{11}(t) = -\infty \qquad t < \tau_2'. \tag{19b}$$

We shall show that the section $\gamma|_{[\tau_1', \tau_2']}$ includes a pair of conjugate points. The components of the curvature tensor are continuous along γ so they are bounded on any closed interval. Hence the functions $R_{i4i4}(t)$ also have lower bounds, $-M_i$, on $[\tau_1', \tau_2']$ where the M_i are positive numbers. Consider now the equations of (2)

$$\frac{\mathrm{d}\theta_{ii}}{\mathrm{d}t} = -R_{i4i4} - \theta_{ii}^2 - \sum_{k=1, k \neq i}^{n} \theta_{ik}^2. \tag{20}$$

If we take the above bounds into account, we get that

$$\frac{\mathrm{d}\theta_{ii}}{\mathrm{d}t} \leqslant M_i - \theta_{ii}^2 - \sum_{k=1, k \neq i}^{n} \theta_{ik}^2 < M_i \tag{21}$$

for all possible values of the index i. After integrating this equation we find that

$$\theta_{ii}(t) \leqslant M_i(t-t_0)+\theta_{ii}(t_0) \qquad t \geqslant t_0 \tag{22a}$$

$$\theta_{ii}(t) \geqslant M_i(t-t_0)+\theta_{ii}(t_0) \qquad t \leqslant t_0. \tag{22b}$$

Define the barrier points α_i, β_i, γ_i and δ_i of the domain of the functions $\theta_{ii}(t)$ as follows:

$$\alpha_i < t_0: \lim_{t\to\alpha_i} \theta_{ii}(t) = -\infty \qquad t > \alpha_i \tag{23a}$$

$$\beta_i < t_0: \lim_{t\to\beta_i} \theta_{ii}(t) = +\infty \qquad t > \beta_i \tag{23b}$$

$$\gamma_i > t_0: \lim_{t\to\gamma_i} \theta_{ii}(t) = -\infty \qquad t < \gamma_i \tag{23c}$$

$$\delta_i > t_0: \lim_{t\to\delta_i} \theta_{ii}(t) = +\infty \qquad t < \delta_i. \tag{23d}$$

It may happen that some of the barrier points do not exist. As a consequence of the inequalities (22) we have that $\alpha_i = -\infty$, and $\delta_i = +\infty$. Now, let τ_1 and τ_2 be defined by $\tau_1 := \max\{\beta_i, \tau_1'\}$ and $\tau_2 := \min\{\gamma_i, \tau_2'\}$. Then it can easily be seen that $\theta(t) = \Sigma_{i=1}^n \theta_{ii}(t)$ is continuous on (τ_1, τ_2), furthermore the conditions (18a) and (18b) are satisfied. In addition we have that $[\tau_1, \tau_2] \subset [\tau_1', \tau_2']$ holds. Consequently the points $\gamma(\tau_1)$ and $\gamma(\tau_2)$ are conjugate to each other along γ between the points $\gamma(\tau_1')$ and $\gamma(\tau_2')$. □

One can find an argument in [5] that the averaged weak energy condition is satisfied by any physically reasonable quantum stress–energy tensor. However, as yet this is only a hypothesis because the author has proved it for special scalar fields on two-dimensional spacetimes, only. Furthermore, most of the singularity theorems require convergence conditions for timelike geodesics as well. Thus even if the above hypothesis were true it would not cover all physically reasonable cases. Thus it is interesting not only in its own right to look for weaker conditions assuring the existence of conjugate points along causal geodesics. Borde has continued the work started by Tipler and has given a weaker integral energy condition.

The present approach differs from the earlier ones by involving restrictions only on one of the diagonal components of the tidal-force tensor, consequently the trace $R_{ab}T^aT^b$ can be arbitrary along the examined causal geodesic.

The tidal-force tensor, like the curvature one, is divided into two parts, the Weyl and the Ricci parts. In the earlier results on the focusing of causal congruences some conditions were imposed on the Ricci part. Hence, it is very important to note that there can exist such spacetimes where the conditions imposed on the $R_{ab}T^aT^b$ are not satisfied, and at the same time our condition given in terms of the tidal curvature (including the Weyl part) holds. One can generate such spacetimes, for example, by certain perturbations of the homogeneous and isotropic cosmologies filled by some matter violating the energy conditions.

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